Chapter 2.1 Projectile Range

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/projectiles/projectile-range/

What this chapter uses

This chapter uses two results of the previous part. (1.2.1) gives the Velocity under a constant acceleration. (1.1.7) gives the position under a constant acceleration.

2.1.1 From launch to landing

Assumptions

This section uses two assumptions. First, the projectile lands at the height from which it was launched (level ground). Second, air resistance is zero.

The launch velocity is stated in polar form: a Launch speed and a Launch angle above the horizontal. (1.1.3) converts the launch velocity into a horizontal component $v_0 \cos(\theta)$ and a vertical component $v_0 \sin(\theta)$. The Gravitational acceleration is vertical. So the horizontal velocity component is constant during the flight. The vertical velocity component decreases, is zero at the highest point, and is negative after the highest point. So the Flight angle φ falls from θ at launch to 0° at the highest point and to $-\theta$ at landing. Diagram 2.1.1 shows the steps of the derivation:

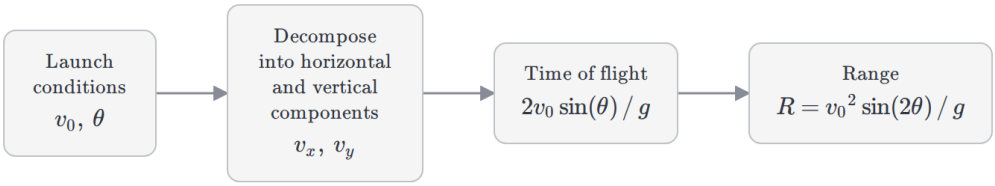


Diagram 2.1.1 Steps of the range derivation. From the launch conditions to the range, through the two velocity components and the time of flight.

The time of flight is the time from launch to landing. The vertical position is zero at launch and at landing. Setting the vertical position to zero and solving for the time gives a time of flight of $2v_0 \sin(\theta)/g$. The Range is the horizontal distance from the launch point to the landing point. For a constant horizontal velocity component, the range is that component multiplied by the time of flight. Substituting both expressions gives the range:

$$R = v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g} \tag{2.1.1}$$

$$= \frac{{v_0}^2 \sin(2\theta)}{g} \quad \text{double-angle identity}$$

Solving (2.1.1) for the Launch speed gives the launch speed that produces a given range. Three symbols of the result are under the square root sign, and each of them still opens its explanation:

$$v_0 = \sqrt{R\, g \csc(2\theta)} \tag{2.1.2}$$

Equation (2.1.2) uses $\csc(2\theta) = 1/\sin(2\theta)$. It is defined only for $\sin(2\theta) > 0$, i.e., for $0° < \theta < 90°$.

Caution

(2.1.1) is correct only if the launch point and the landing point are at the same height and air resistance is zero [1].

2.1.2 Launching from a height

If the launch point is higher than the landing point, (2.1.1) does not apply. Let Launch height be the height of the launch point above the landing plane. The time of flight is then the positive solution of a quadratic equation for the vertical position [2]. The range is the horizontal velocity component multiplied by this time of flight:

$$R = \frac{v_0 \cos(\theta)}{g}\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2gh}\right] \tag{2.1.3}$$

For Launch height $= 0$, the square root in (2.1.3) equals $v_0 \sin(\theta)$. So the bracket equals $2v_0 \sin(\theta)$, and (2.1.3) reduces to (2.1.1).

(2.1.3) also shows how this engine sizes delimiters. Three scalable pairs are nested: the parentheses of the squared group are inside the square root, and the square root is inside the outer bracket. The square root is as tall as a fraction, and the outer bracket must be taller than the square root. Stock KaTeX lets a `\left … \right` pair be slightly shorter than its content. Then all three pairs get the same glyph, and the reader cannot see the nesting. This engine uses a patched KaTeX that makes every pair strictly taller than its content. So each level is visibly larger than the level inside it, and the source contains no manual size command such as `\Bigl` or `\biggl`.

2.1.3 Worked example: try it

The explorer in Figure 2.1.1 evaluates (2.1.1) for a launch speed and a launch angle that you choose. If (2.1.1) is on the screen, a reference to it highlights the equation. If the equation is off the screen, the reference opens a preview of it.

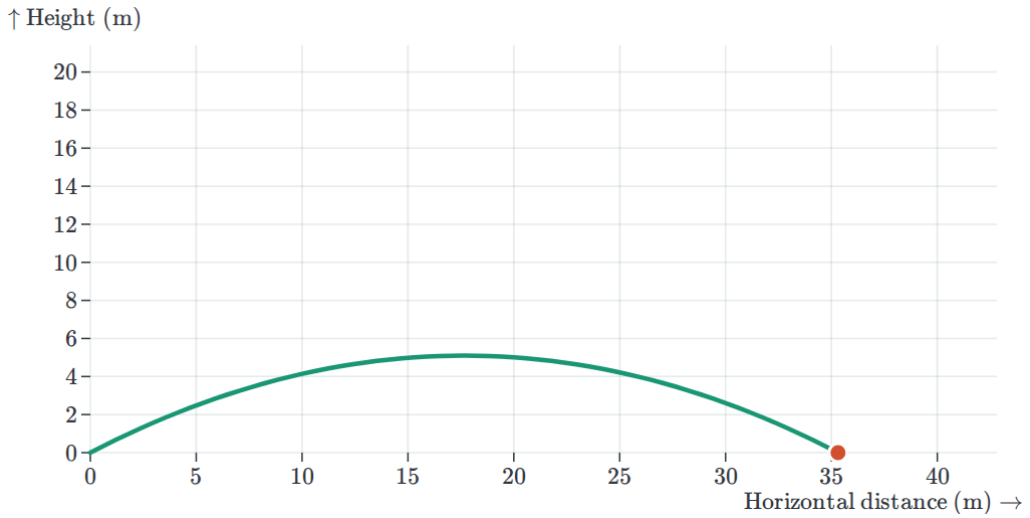


Figure 2.1.1 Projectile trajectory explorer. The path from launch to landing; the dot marks the landing point. Shown for $v_0 = 20$ m/s and $\theta = 30°$: range 35.3 m, time of flight 2.04 s.

For a fixed launch speed, a larger launch angle gives a higher trajectory, but not always a larger range. The range is the horizontal velocity component multiplied by the time of flight. A larger launch angle increases the time of flight but decreases the horizontal velocity component.

The next chapter plots the range against the launch angle for one launch speed, in Figure 2.3.1. That plot shows which launch angle gives the largest range.

A reference to Figure 2.1.1 scrolls back to the explorer.

2.1.4 Sampled flight data

The explorer shows one trajectory at a time. Table 2.1.1 instead lists the state of one flight at eleven values of Time, from launch to landing. The flight has a Launch speed of 45 m/s and a Launch angle of 55°. Each row gives the Position by its Horizontal coordinate and its Vertical coordinate, the Velocity, and the Acceleration from the constant-acceleration equations of the previous part. Each row also gives the Kinetic energy and the Potential energy per unit Mass:

t (s)	x (m)	y (m)	v_x (m/s)	v_y (m/s)	a_x (m/s^2)	a_y (m/s^2)	$\lvert\boldsymbol{v}\rvert$ (m/s)	θ (°)	KE/m (J/kg)	PE/m (J/kg)
0.00	0.00	0.00	25.81	36.86	0.00	-9.80	45.00	55.00	1012.50	0.00
0.75	19.36	24.89	25.81	29.51	0.00	-9.80	39.21	48.83	768.58	243.92
1.50	38.72	44.27	25.81	22.16	0.00	-9.80	34.02	40.65	578.68	433.82
2.25	58.07	58.13	25.81	14.81	0.00	-9.80	29.76	29.85	442.80	569.70
3.00	77.43	66.49	25.81	7.46	0.00	-9.80	26.87	16.12	360.94	651.56
3.76	97.05	69.33	25.81	0.01	0.00	-9.80	25.81	0.03	333.10	679.40
4.50	116.15	66.65	25.81	-7.24	0.00	-9.80	26.81	-15.67	359.30	653.20
5.25	135.51	58.47	25.81	-14.59	0.00	-9.80	29.65	-29.47	439.51	572.99
6.00	154.87	44.77	25.81	-21.94	0.00	-9.80	33.87	-40.36	573.74	438.76
6.75	174.22	25.56	25.81	-29.29	0.00	-9.80	39.04	-48.61	762.00	250.50
7.52	194.17	0.00	25.81	-36.86	0.00	-9.80	45.00	-55.00	1012.50	0.00

Table 2.1.1 Sampled flight data. Position, velocity, acceleration and specific mechanical energy through one flight, from launch to landing.

Eleven columns are enough to make Table 2.1.1 scroll horizontally on most screens. Each row agrees with the results of this chapter:

- The Gravitational acceleration appears only in the a_y column. The a_x column is zero in every row.
- The v_x column has the same value in every row.
- The sum of the last two columns is 1012.50 J/kg in every row. This sum is the mechanical energy per unit Mass. The mechanical energy is constant because air resistance is zero.
- At $t = 3.76$ s, the v_y column is approximately zero. So this row is the highest point of the flight.
- The last row is the landing. For this launch speed and launch angle, (2.1.1) gives $R \approx 194$ m, and the x column gives 194.17 m.

Knowledge check 2.1.1: Projectile range

1. A ball is launched over level ground at 20 m/s, 30° above the horizontal. Using $g = 9.81$ m/s^2 and ignoring air resistance, what is its range in meters?

Answers: end of the chapter.

References

- [1] Ling et al., University Physics Volume 1 (2016). ch. 4, "Projectile Motion". https://openstax.org/details/books/university-physics-volume-1
- [2] Urone & Hinrichs, College Physics 2e (2022). ch. 3, "Projectile Motion", Example 3.5. https://openstax.org/details/books/college-physics-2e

Notation

φ **Flight angle**. The angle of the projectile's velocity above the horizontal at one moment of the flight. It equals the launch angle at launch, is zero at the highest point, and is the negative of the launch angle at landing on level ground. *Units:* degrees above the horizontal.

a_x **Horizontal acceleration**. The acceleration along the horizontal axis. Without air resistance nothing pushes the projectile sideways, so it is zero for the whole flight. *Units:* meters per second squared.

$\square_x$ **Horizontal component**. The part of a vector quantity along the horizontal axis, positive in the direction of flight.

KE **Kinetic energy**. The energy of the projectile due to its motion. It equals one half of the mass multiplied by the square of the speed. *Units:* joules.

h **Launch height**. The height of the launch point above the plane on which the projectile lands. It is zero for level ground. *Units:* meters.

m **Mass**. The projectile's mass. The sampled flight data table gives the kinetic and the potential energy per unit mass, so the mass does not appear in any computed value. *Units:* kilograms.

PE **Potential energy**. The energy of the projectile due to its height above the launch point in the gravitational field. *Units:* joules.

a_y **Vertical acceleration**. The acceleration along the vertical axis. Gravity alone acts on the projectile, so it equals the negative of the gravitational acceleration for the whole flight. *Units:* meters per second squared.

$\square_y$ **Vertical component**. The part of a vector quantity along the vertical axis, positive upward.

x **Horizontal coordinate**. The signed distance from the origin along the horizontal axis, measured positive to the right. *Units:* meters.

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

$\boldsymbol{a}$ **Acceleration**. The rate at which velocity changes with time. In this course it is held constant over a scenario, so it is the same vector at every instant. *Units:* meters per second squared.

$$\boldsymbol{a} = \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}t}$$

g **Gravitational acceleration**. The magnitude of gravitational acceleration, treated as constant near the ground. *Units:* meters per second squared.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

$\boldsymbol{r}$ **Position**. The object's location at elapsed time t, taken as a vector from a fixed origin so a two-dimensional motion keeps its horizontal and vertical parts separate. *Units:* meters.

$$\boldsymbol{r} = (x,\, y)$$

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters.

$$R = \frac{{v_0}^2 \sin(2\theta)}{g}$$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second.

$$\boldsymbol{v} = \frac{\mathrm{d}\,\boldsymbol{r}}{\mathrm{d}t}\,; \quad v(t) = v_0 + a\,t$$

Answers to the knowledge checks

Knowledge check 2.1.1: Projectile range

1. **35.3** (accepted within ±0.5). $R = \dfrac{{v_0}^2 \sin(2\theta)}{g} = \dfrac{(20\ \mathrm{m/s})^2 \times \sin(60^\circ)}{9.81\ \mathrm{m/s^2}} = \dfrac{400\ \mathrm{m^2/s^2} \times 0.8660}{9.81\ \mathrm{m/s^2}} \approx 35.3\ \mathrm{m}$.