Projectile Range

Explico Playground — Motion

Course Author

Content unchanged since version 1.0.0

From launch to landing

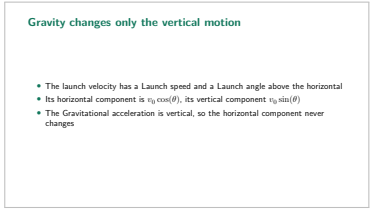
Gravity changes only the vertical motion

- The launch velocity has a Launch speed and a Launch angle above the horizontal
- Its horizontal component is $v_0 \cos(\theta)$, its vertical component $v_0 \sin(\theta)$
- The Gravitational acceleration is vertical, so the horizontal component never changes

v_0 Launch speed θ Launch angle g Gravitational acceleration

Slide 3

Gravity changes only the vertical motion



Start from the two numbers a thrower controls: how fast and at what angle. Split the launch velocity into its two components before anything else. Then ask the room which component gravity changes. The answer, only the vertical one, is the whole idea of this chapter.

The range takes three steps to derive

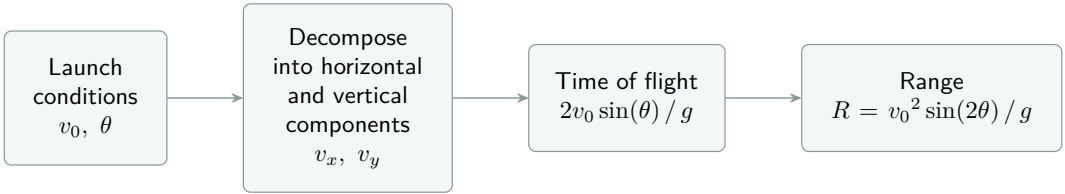


Diagram 2.1.1 **Steps of the range derivation.** From the launch conditions to the range, through the two velocity components and the time of flight.

Slide 4

The range takes three steps to derive

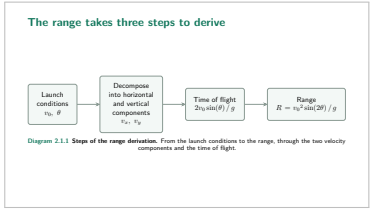


Diagram 2.1.1 **Steps of the range derivation.** From the launch conditions to the range, through the two velocity components and the time of flight.

Walk the diagram from left to right: the launch conditions, the two components, the time of flight, and the range. The next two slides do the two middle steps.

The flight ends when the height is zero again

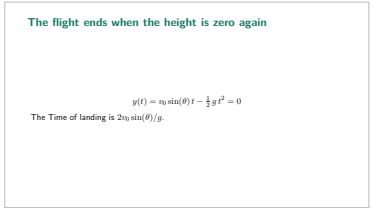
$$y(t) = v_0 \sin(\theta)\, t - \tfrac{1}{2}\, g\, t^2 = 0$$

The Time of landing is $2v_0 \sin(\theta)/g$.

y Vertical coordinate t Time

Slide 5

The flight ends when the height is zero again



The vertical position is zero at launch and at landing. Factor out the time: one root is the launch, the other is the landing. Point at the factor two: the climb and the fall take equally long.

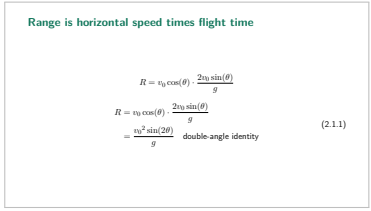
Range is horizontal speed times flight time

$$R = v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g}$$

$$\begin{aligned} R &= v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g} \\ &= \frac{{v_0}^2 \sin(2\theta)}{g} \quad \text{double-angle identity} \end{aligned} \tag{2.1.1}$$

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Range is horizontal speed times flight time



The horizontal speed is constant, so the Range is that speed multiplied by the time of flight from the previous slide. The double-angle identity turns the product of sine and cosine into one sine. This is the range formula of the notes.

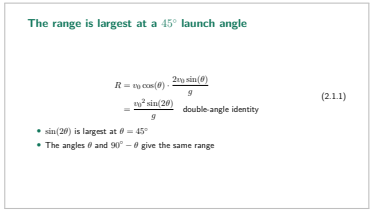
The range is largest at a 45° launch angle

$$
\begin{aligned}
R &= v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g} \\
&= \frac{{v_0}^2 \sin(2\theta)}{g} \quad \text{double-angle identity}
\end{aligned}
\tag{2.1.1}
$$

- $\sin(2\theta)$ is largest at $\theta = 45^\circ$
- The angles θ and $90^\circ - \theta$ give the same range

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The range is largest at a 45° launch angle



Ask where the sine of twice the angle is largest before showing the answer; many in the room say 90°. The second point follows from the same sine: two complementary angles give the same value, so one shallow and one steep throw land at the same place.

Try it

Watch the range as the angle sweeps

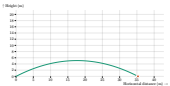
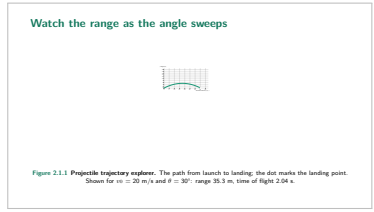


Figure 2.1.1 **Projectile trajectory explorer.** The path from launch to landing; the dot marks the landing point. Shown for $v_0 = 20$ m/s and $\theta = 30°$: range 35.3 m, time of flight 2.04 s.

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Watch the range as the angle sweeps



Let the angle sweep from $15°$ to $75°$ and watch the landing point: it moves out until $45°$ and comes back after. At home, move the slider yourself and read the range off the caption.

From a height

A launch from a height flies farther

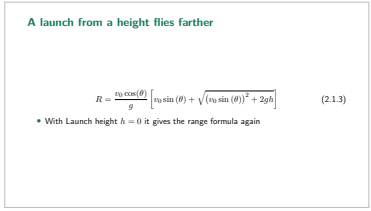
$$R = \frac{v_0 \cos(\theta)}{g} \left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2gh} \right] \tag{2.1.3}$$

- With Launch height $h = 0$ it gives the range formula again

h Launch height

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A launch from a height flies farther



From a cliff or a tower the landing is lower than the launch, and the range formula no longer applies. The time of flight is the positive root of a quadratic equation. Show that the bracket becomes twice the vertical component when the height is zero.

Check

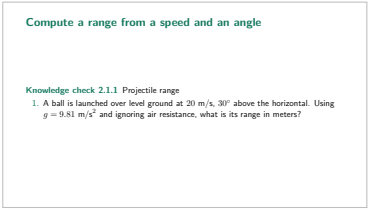
Compute a range from a speed and an angle

Knowledge check 2.1.1 Projectile range

1. A ball is launched over level ground at 20 m/s, 30° above the horizontal. Using $g = 9.81$ m/s^2 and ignoring air resistance, what is its range in meters?

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Compute a range from a speed and an angle



Answer: 35.3 (within ± 0.5).

$$R = \frac{{v_0}^2 \sin(2\theta)}{g} = \frac{(20\ \text{m/s})^2 \times \sin(60^\circ)}{9.81\ \text{m/s}^2} = \frac{400\ \text{m}^2/\text{s}^2 \times 0.8660}{9.81\ \text{m/s}^2} \approx 35.3\ \text{m}.$$

What you can now do

- Given a launch speed and angle over level ground, compute the horizontal range of a projectile.

Notation

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

g **Gravitational acceleration**. The magnitude of gravitational acceleration, treated as constant near the ground. *Units:* meters per second squared.

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second. $\mathbf{v} = \dfrac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}t}\,; \quad v(t) = v_0 + a\,t$

$\square_x$ **Horizontal component**. The part of a vector quantity along the horizontal axis, positive in the direction of flight.

Notation (continued)

$\square_y$ **Vertical component**. The part of a vector quantity along the vertical axis, positive upward.

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters. $R = \dfrac{{v_0}^2 \sin(2\theta)}{g}$

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

h **Launch height**. The height of the launch point above the plane on which the projectile lands. It is zero for level ground. *Units:* meters.