Chapter 2.3 The Optimal Launch Angle

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/projectiles/optimal-launch-angle/

By (2.1.1), the Range depends on the Launch angle only through $\sin(2\theta)$. The previous chapter derives that equation. For $0^\circ \le \theta \le 90^\circ$, the argument 2θ is between 0° and 180°. On this interval, the sine has its maximum value, 1, at 90°. So for a fixed launch speed, the range is largest at $2\theta = 90^\circ$, i.e., at $\theta = 45^\circ$. The explorer in Figure 2.1.1 shows this result. Diagram 2.3.1 shows the inputs to the search for this launch angle.

2.3.1 A polynomial maximum, no calculus

This section finds the same launch angle as the maximum of a polynomial. For $0^\circ \le \theta \le 90^\circ$, the product $\sin(\theta)\cos(\theta)$ is not negative. So the range is largest where the square of this product is largest. Let u be the Squared sine of the launch angle: $u = \sin^2(\theta)$. Rewriting the square of the product in terms of u gives a second-order polynomial:

$$\begin{aligned} \sin^2(\theta)\cos^2(\theta) &= \sin^2(\theta)(1-\sin^2(\theta)) \quad \text{Pythagorean identity} \\ &= u(1-u) \quad \text{definition of } u \\ &= u - u^2 \end{aligned} \tag{2.3.1}$$

The graph of this polynomial is a downward parabola with roots at $u = 0$ and $u = 1$. A downward parabola has its maximum halfway between its roots, so the polynomial is largest at $u = \frac{1}{2}$. Converting this value of u back to a launch angle gives the Optimal launch angle:

$$\begin{aligned} \theta^* &= \arcsin\left(\sqrt{u}\right) \qquad \text{definition of } u \text{, with } \sin(\theta) \ge 0 \\ &= \arcsin\left(\tfrac{1}{\sqrt{2}}\right) \qquad \text{vertex at } u = \tfrac{1}{2} \\ &= 45^\circ \end{aligned} \tag{2.3.2}$$

Setting the derivative of the polynomial to zero gives the same vertex (show how). But the symmetry of the parabola gives the vertex without a derivative. In (2.1.1), the launch speed appears only in the factor ${v_0}^2$, which does not depend on the launch angle. A positive constant factor does not change the position of the maximum. So the Optimal launch angle is 45° for every launch speed. [a] checks the vertex numerically and graphically.

Show the full calculation using calculus.

Take the derivative of the expanded polynomial and set it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}u}(u - u^2) = 1 - 2u = 0 \tag{2.3.3}$$

The solution of this equation is $u = \frac{1}{2}$.

2.3.2 Complementary launch angles give equal ranges

Two launch angles are complementary if their sum is 90°. The Complementary launch angle of θ is $\theta' = 90^\circ - \theta$. On this page the prime marks this angle and does not mean a derivative. For complementary launch angles, $\sin(2\theta') = \sin(180^\circ - 2\theta) = \sin(2\theta)$. So two complementary launch angles give equal ranges. For example, the range at a launch angle of 30° equals the

range at a launch angle of 60°. The Optimal launch angle is the only launch angle that is its own complement.

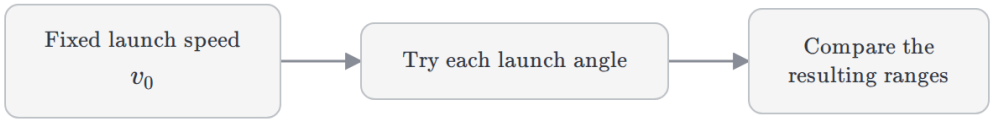


Diagram 2.3.1 Inputs to the launch-angle search. The launch speed stays fixed while each launch angle is tried in turn, and the ranges they give are compared.

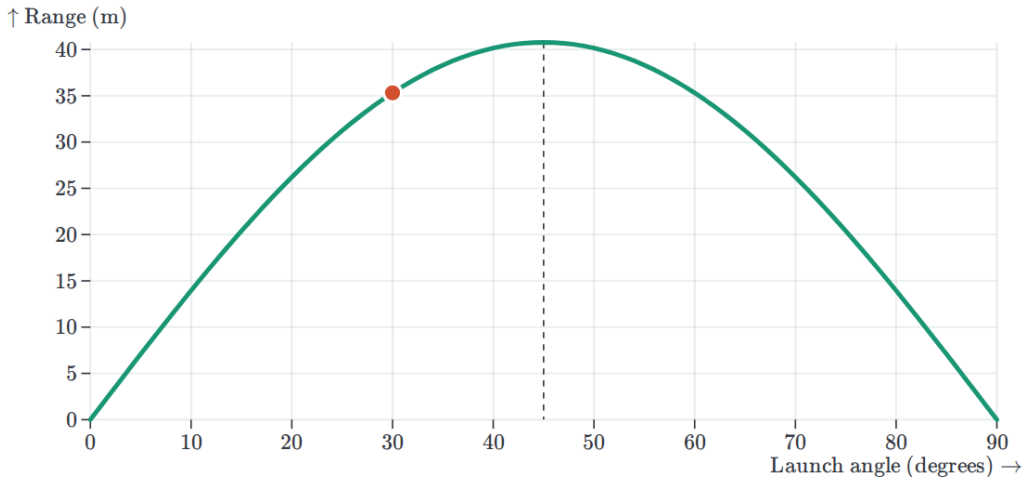


Figure 2.3.1 Range vs. launch angle explorer. The range at every launch angle for one launch speed; the dashed line marks 45°, where the range is largest. Shown for $v_0 = 20$ m/s; the dot marks $\theta = 30°$: range 35.3 m, 5.5 m short of the maximum (40.8 m, at 45°).

The curve is symmetric about 45°. A launch angle of 20° and a launch angle of 70° give the same range. The trajectory at 20° is lower and has a shorter time of flight than the trajectory at 70°.

Memo a: Checking the polynomial vertex: its values on both sides of the vertex, and its downward-parabola shape

Evaluating the polynomial at the vertex, $u = \frac{1}{2}$, gives its maximum value:

$$u(1-u)\big|_{u=1/2} = \tfrac{1}{4} \tag{2.3.4}$$

Two values of u at equal distances from the vertex give equal values of the polynomial, and both values are smaller than the maximum.

Squared sine of the launch angle	Value of the polynomial
0	0
$\frac{1}{4}$	$\frac{3}{16}$
$\frac{1}{2}$	$\frac{1}{4}$
$\frac{3}{4}$	$\frac{3}{16}$
1	0

Table 2.3.1 Polynomial values around the vertex. The polynomial at five inputs placed symmetrically about the vertex, 1/2: inputs equally far from it give equal values.

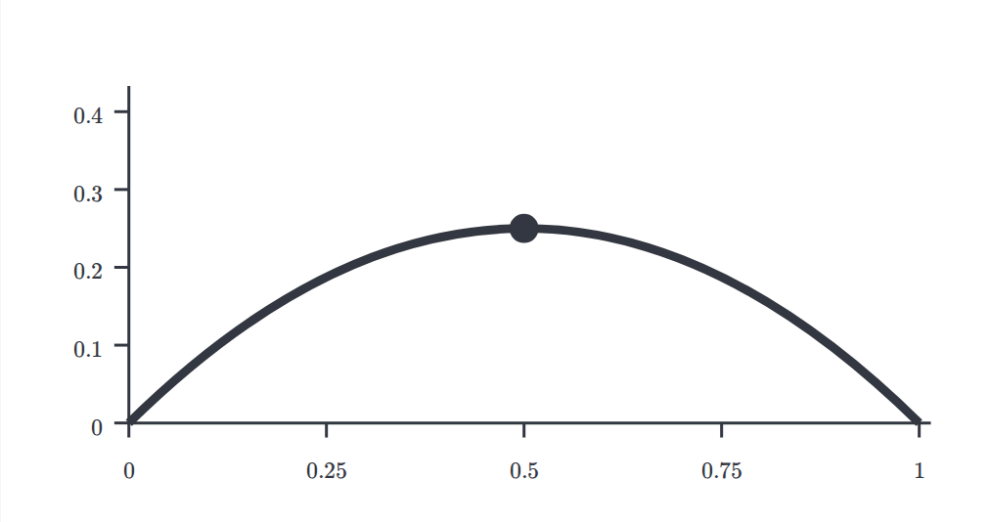


Figure 2.3.2 The polynomial is a downward parabola. From 0 at input 0 it rises to its maximum, 0.25, at input 0.5, and falls symmetrically back to 0 at input 1.

(2.3.4), Table 2.3.1, and Figure 2.3.2 are inside the memo. A reference to each of them also works while the memo is closed.

Knowledge check 2.3.1: Optimal launch angle

1. For a fixed launch speed over level ground, with no air resistance, which launch angle maximizes range?
   - (a) 30°
   - (b) 45°
   - (c) 60°
   - (d) 90°

Answers: end of the chapter.

Notation

On this page, $\square'$ is not an operator. The prime marks the complementary launch angle, not a derivative.

θ' **Complementary launch angle**. The launch angle whose sum with a given launch angle is 90 degrees. *Units:* degrees above the horizontal.

θ^* **Optimal launch angle**. The launch angle that maximizes range for a given fixed launch speed, over level ground. *Units:* degrees above the horizontal.

u **Squared sine of the launch angle**. The squared sine of the launch angle, used as the variable in the range polynomial. *Units:* dimensionless.

$$u = \sin^2(\theta)$$

g **Gravitational acceleration**. The magnitude of gravitational acceleration, treated as constant near the ground. *Units:* meters per second squared.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters.

$$R = \frac{{v_0}^2 \sin(2\theta)}{g}$$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

Answers to the knowledge checks

Knowledge check 2.3.1: Optimal launch angle

1. **(b)** $45°$**.** The range is proportional to $\sin(2\theta)$. For launch angles from $0°$ to $90°$, $\sin(2\theta)$ has its maximum at $2\theta = 90°$. So $\theta^* = 45°$.