The Optimal Launch Angle

Explico Playground — Motion

Course Author

Content with changes after version 1.0.0, not yet released

The angle that flies farthest

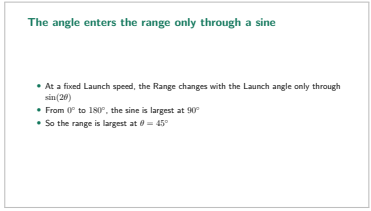
The angle enters the range only through a sine

- At a fixed Launch speed, the Range changes with the Launch angle only through $\sin(2\theta)$
- From 0° to 180°, the sine is largest at 90°
- So the range is largest at $\theta = 45^\circ$

v_0 Launch speed R Range θ Launch angle

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The angle enters the range only through a sine



The range formula of the previous chapter has the launch angle only in the factor $\sin(2\theta)$. For launch angles from 0° to 90°, twice the angle runs from 0° to 180°. On that interval the sine is largest at 90°.

The search fixes the speed and tries each angle

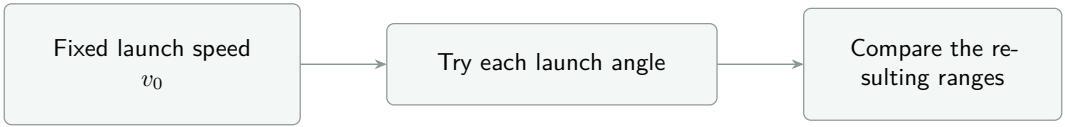


Diagram 2.3.1 **Inputs to the launch-angle search.** The launch speed stays fixed while each launch angle is tried in turn, and the ranges they give are compared.

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The search fixes the speed and tries each angle

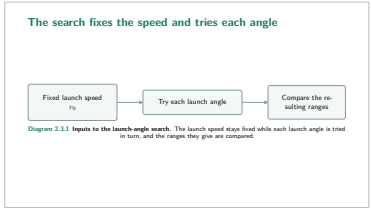


Diagram 2.3.1 **Inputs to the launch-angle search.** The launch speed stays fixed while each launch angle is tried in turn, and the ranges they give are compared.

The launch speed stays fixed while the launch angle varies. Each angle gives a range, and the search compares the ranges.

A maximum without calculus

A square turns the product into a polynomial

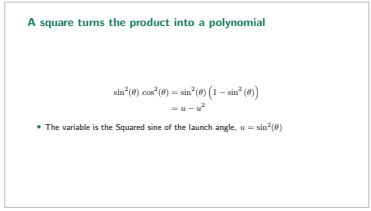
$$\begin{aligned}\sin^2(\theta)\,\cos^2(\theta) &= \sin^2(\theta)\left(1-\sin^2(\theta)\right)\\ &= u - u^2\end{aligned}$$

- The variable is the Squared sine of the launch angle, $u = \sin^2(\theta)$

u Squared sine of the launch angle

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A square turns the product into a polynomial



Between $0°$ and $90°$ the product of the sine and the cosine is not negative, so the range is largest where the square of the product is largest. The Pythagorean identity writes the square of the cosine as one minus the square of the sine. The result is a polynomial of second order.

The parabola peaks halfway between its roots

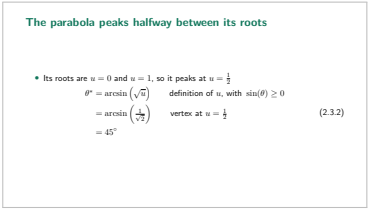
- Its roots are $u = 0$ and $u = 1$, so it peaks at $u = \frac{1}{2}$

$$\begin{aligned} \theta^* &= \arcsin\left(\sqrt{u}\right) && \text{definition of } u, \text{ with } \sin(\theta) \geq 0 \\ &= \arcsin\left(\tfrac{1}{\sqrt{2}}\right) && \text{vertex at } u = \tfrac{1}{2} \\ &= 45^\circ \end{aligned} \tag{2.3.2}$$

θ^* Optimal launch angle

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The parabola peaks halfway between its roots



A downward parabola has its maximum halfway between its roots. Converting $u = \frac{1}{2}$ back to an angle gives the Optimal launch angle. The launch speed is a positive factor in front of the polynomial, so it does not move the maximum.

The polynomial is a downward parabola

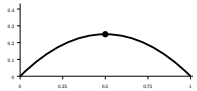


Figure 2.3.2 The polynomial is a downward parabola. From 0 at input 0 it rises to its maximum, 0.25, at input 0.5, and falls symmetrically back to 0 at input 1.

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The polynomial is a downward parabola

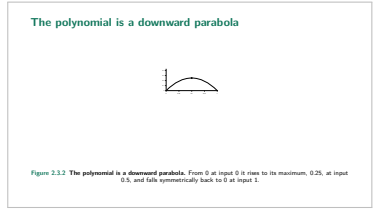


Figure 2.3.2 **The polynomial is a downward parabola.** From 0 at input 0 it rises to its maximum, 0.25, at input 0.5, and falls symmetrically back to 0 at input 1.

The value at the vertex is one quarter. Two inputs at the same distance from the vertex give the same value, and both values are smaller than one quarter.

Complementary angles

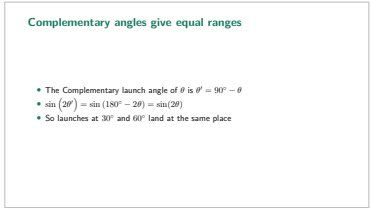
Complementary angles give equal ranges

- The Complementary launch angle of θ is $\theta' = 90^\circ - \theta$
- $\sin\left(2\theta'\right) = \sin\left(180^\circ - 2\theta\right) = \sin(2\theta)$
- So launches at 30° and 60° land at the same place

θ' Complementary launch angle

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Complementary angles give equal ranges



In this lesson the prime marks the complementary angle, not a derivative. The optimal angle is the only launch angle that is its own complement.

The range curve is symmetric about $45°$

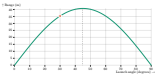


Figure 2.3.1 **Range vs. launch angle explorer.** The range at every launch angle for one launch speed; the dashed line marks 45°, where the range is largest. Shown for $v_0 = 20$ m/s; the dot marks $\theta = 30°$: range 35.3 m, 5.5 m short of the maximum (40.8 m, at 45°).

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The range curve is symmetric about $45°$

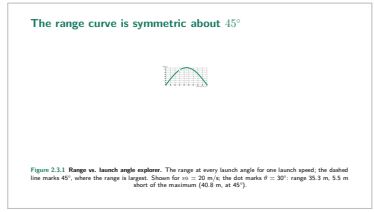


Figure 2.3.1 **Range vs. launch angle explorer.** The range at every launch angle for one launch speed; the dashed line marks 45°, where the range is largest. Shown for $v_0 = 20$ m/s; the dot marks $\theta = 30°$: range 35.3 m, 5.5 m short of the maximum (40.8 m, at 45°).

Let the angle sweep from $15°$ to $75°$ and watch the marker climb the curve to $45°$ and come back down. Launches at $20°$ and at $70°$ land at the same place, but the one at $70°$ flies higher and longer. At home, set the speed and the angle yourself.

Check

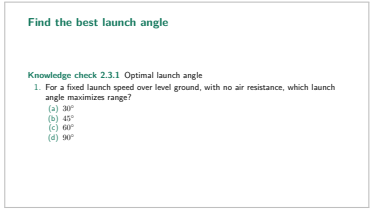
Find the best launch angle

Knowledge check 2.3.1 Optimal launch angle

1. For a fixed launch speed over level ground, with no air resistance, which launch angle maximizes range?
   (a) $30°$
   (b) $45°$
   (c) $60°$
   (d) $90°$

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Find the best launch angle



Answer: (b) $45°$. The range is proportional to $\sin(2\theta)$. For launch angles from $0°$ to $90°$, $\sin(2\theta)$ has its maximum at $2\theta = 90°$. So $\theta^* = 45°$.

What you can now do

- For a fixed launch speed over level ground, identify the launch angle that maximizes range.

Notation

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters. $R = \dfrac{{v_0}^2 \sin(2\theta)}{g}$

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

u **Squared sine of the launch angle**. The squared sine of the launch angle, used as the variable in the range polynomial. *Units:* dimensionless. $u = \sin^2(\theta)$

θ^* **Optimal launch angle**. The launch angle that maximizes range for a given fixed launch speed, over level ground. *Units:* degrees above the horizontal.

Notation (continued)

θ' **Complementary launch angle**. The launch angle whose sum with a given launch angle is 90 degrees. *Units:* degrees above the horizontal.