Chapter 2.3

The Optimal Launch Angle

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: `https://danieltuzes.github.io/explico/playground/projectiles/optimal-launch-angle/`

By (2.1.1), the Range depends on the Launch angle only through $\sin(2\theta)$. The previous chapter derives that equation. For $0° \le \theta \le 90°$, the argument 2θ is between 0° and 180°. On this interval, the sine has its maximum value, 1, at 90°. So for a fixed launch speed, the range is largest at $2\theta = 90°$, i.e., at $\theta = 45°$. The explorer in Figure 2.1.1 shows this result. Diagram 2.3.1 shows the inputs to the search for this launch angle.

2.3.1 A polynomial maximum, no calculus

This section finds the same launch angle as the maximum of a polynomial. For $0° \le \theta \le 90°$, the product $\sin(\theta)\cos(\theta)$ is not negative. So the range is largest where the square of this product is largest. Let u be the Squared sine of the launch angle: $u = \sin^2(\theta)$. Rewriting the square of the product in terms of u gives a second-order polynomial:

$$\begin{aligned} \sin^2(\theta)\,\cos^2(\theta) &= \sin^2(\theta)\left(1-\sin^2(\theta)\right) && \text{Pythagorean identity} \\ &= u\,(1-u) && \text{definition of } u \\ &= u - u^2 \end{aligned} \tag{2.3.1}$$

The graph of this polynomial is a downward parabola with roots at $u = 0$ and $u = 1$. A downward parabola has its maximum halfway between its roots, so the polynomial is largest at $u = \frac{1}{2}$. Converting this value of u back to a launch angle gives the Optimal launch angle:

$$\begin{aligned} \theta^* &= \arcsin\left(\sqrt{u}\right) && \text{definition of } u, \text{ with } \sin(\theta) \ge 0 \\ &= \arcsin\left(\tfrac{1}{\sqrt{2}}\right) && \text{vertex at } u = \tfrac{1}{2} \\ &= 45° \end{aligned} \tag{2.3.2}$$

Setting the derivative of the polynomial to zero gives the same vertex (show how). But the symmetry of the parabola gives the vertex without a derivative. In (2.1.1), the launch speed appears only in the factor ${v_0}^2$, which does not depend on the launch angle. A positive constant factor does not change the position of the maximum. So the Optimal launch angle is 45° for every launch speed. [a] checks the vertex numerically and graphically.

Show the full calculation using calculus.

Take the derivative of the expanded polynomial and set it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u - u^2\right) = 1 - 2u = 0 \tag{2.3.3}$$

The solution of this equation is $u = \frac{1}{2}$.

2.3.2 Complementary launch angles give equal ranges

Two launch angles are complementary if their sum is 90°. The Complementary launch angle of θ is $\theta' = 90° - \theta$. On this page the prime marks this angle and does not mean a derivative. For complementary launch angles, $\sin(2\theta') = \sin(180° - 2\theta) = \sin(2\theta)$. So two complementary launch angles give equal ranges. For example, the range at a launch angle of 30° equals the range at a launch angle of 60°. The Optimal launch angle is the only launch angle that is its own complement.

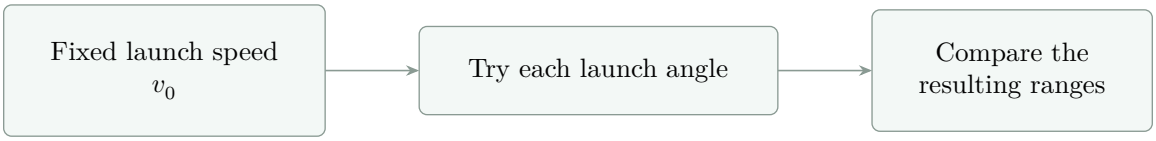


Diagram 2.3.1 **Inputs to the launch-angle search.** The launch speed stays fixed while each launch angle is tried in turn, and the ranges they give are compared.

The curve is symmetric about 45°. A launch angle of 20° and a launch angle of 70° give the same range. The trajectory at 20° is lower and has a shorter time of flight than the trajectory at 70°.

Memo a: Checking the polynomial vertex: its values on both sides of the vertex, and its downward-parabola shape

Evaluating the polynomial at the vertex, $u = \frac{1}{2}$, gives its maximum value:

$$u(1-u)\Big|_{u=1/2} = \frac{1}{4} \tag{2.3.4}$$

Two values of u at equal distances from the vertex give equal values of the polynomial, and both values are smaller than the maximum.

Table 2.3.1 **Polynomial values around the vertex.** The polynomial at five inputs placed symmetrically about the vertex, 1/2: inputs equally far from it give equal values.

Squared sine of the launch angle	Value of the polynomial
0	0
$\frac{1}{4}$	$\frac{3}{16}$
$\frac{1}{2}$	$\frac{1}{4}$
$\frac{3}{4}$	$\frac{3}{16}$
1	0

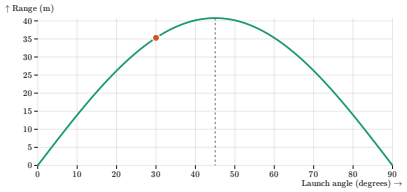


Figure 2.3.1 **Range vs. launch angle explorer.** The range at every launch angle for one launch speed; the dashed line marks 45°, where the range is largest. Shown for $v_0 = 20$ m/s; the dot marks $\theta = 30°$: range 35.3 m, 5.5 m short of the maximum (40.8 m, at 45°).

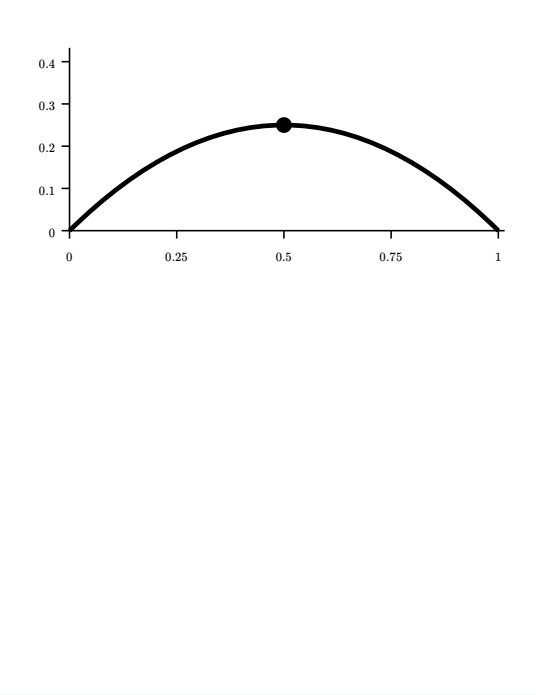


Figure 2.3.2 **The polynomial is a downward parabola.** From 0 at input 0 it rises to its maximum, 0.25, at input 0.5, and falls symmetrically back to 0 at input 1.

(2.3.4), Table 2.3.1, and Figure 2.3.2 are inside the memo. A reference to each of them also works while the memo is closed.

Knowledge check 2.3.1: Optimal launch angle

1. For a fixed launch speed over level ground, with no air resistance, which launch angle maximizes range?

   (a) 30°

   (b) 45°

   (c) 60°

   (d) 90°

Answers: end of the chapter.

Notation

On this page, $\square'$ is not an operator. The prime marks the complementary launch angle, not a derivative.

θ' **Complementary launch angle**. The launch angle whose sum with a given launch angle is 90 degrees. *Units:* degrees above the horizontal.

θ^* **Optimal launch angle**. The launch angle that maximizes range for a given fixed launch speed, over level ground. *Units:* degrees above the horizontal.

u **Squared sine of the launch angle**. The squared sine of the launch angle, used as the variable in the range polynomial. *Units:* dimensionless.

$$u = \sin^2(\theta)$$

g **Gravitational acceleration**. The magnitude of gravitational acceleration, treated as constant near the ground. *Units:* meters per second squared.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters.

$$R = \frac{{v_0}^2 \sin(2\theta)}{g}$$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

Answers to the knowledge checks

Knowledge check 2.3.1: Optimal launch angle

1. **(b)** 45°. The range is proportional to $\sin(2\theta)$. For launch angles from 0° to 90°, $\sin(2\theta)$ has its maximum at $2\theta = 90°$. So $\theta^* = 45°$.