Projectile motion

Course Author

Part 2: Projectile motion

Chapter 2.1 Projectile Range

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/projectiles/projectile-range/

What this chapter uses

This chapter uses two results of the previous part. (1.2.1) gives the Velocity under a constant acceleration. (1.1.7) gives the position under a constant acceleration.

2.1.1 From launch to landing

Assumptions

This section uses two assumptions. First, the projectile lands at the height from which it was launched (level ground). Second, air resistance is zero.

The launch velocity is stated in polar form: a Launch speed and a Launch angle above the horizontal. (1.1.3) converts the launch velocity into a horizontal component $v_0 \cos(\theta)$ and a vertical component $v_0 \sin(\theta)$. The Gravitational acceleration is vertical. So the horizontal velocity component is constant during the flight. The vertical velocity component decreases, is zero at the highest point, and is negative after the highest point. So the Flight angle φ falls from θ at launch to 0° at the highest point and to $-\theta$ at landing. Diagram 2.1.1 shows the steps of the derivation:

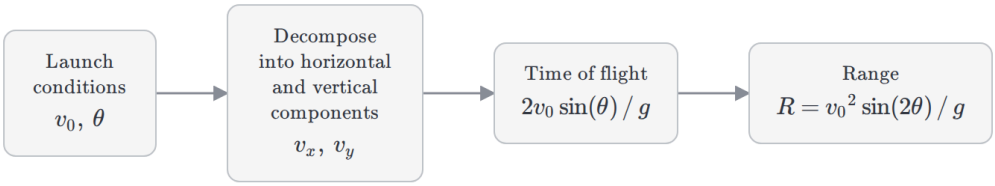


Diagram 2.1.1 Steps of the range derivation. From the launch conditions to the range, through the two velocity components and the time of flight.

The time of flight is the time from launch to landing. The vertical position is zero at launch and at landing. Setting the vertical position to zero and solving for the time gives a time of flight of $2v_0 \sin(\theta)/g$. The Range is the horizontal distance from the launch point to the landing point. For a constant horizontal velocity component, the range is that component multiplied by the time of flight. Substituting both expressions gives the range:

$$R = v_0 \cos(\theta) \cdot \frac{2v_0 \sin(\theta)}{g} \tag{2.1.1}$$

$$= \frac{{v_0}^2 \sin(2\theta)}{g} \quad \text{double-angle identity}$$

Solving (2.1.1) for the Launch speed gives the launch speed that produces a given range. Three symbols of the result are under the square root sign, and each of them still opens its explanation:

$$v_0 = \sqrt{R\, g \csc(2\theta)} \tag{2.1.2}$$

Equation (2.1.2) uses $\csc(2\theta) = 1/\sin(2\theta)$. It is defined only for $\sin(2\theta) > 0$, i.e., for $0° < \theta < 90°$.

Caution

(2.1.1) is correct only if the launch point and the landing point are at the same height and air resistance is zero [1].

2.1.2 Launching from a height

If the launch point is higher than the landing point, (2.1.1) does not apply. Let Launch height be the height of the launch point above the landing plane. The time of flight is then the positive solution of a quadratic equation for the vertical position [2]. The range is the horizontal velocity component multiplied by this time of flight:

$$R = \frac{v_0 \cos(\theta)}{g}\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2gh}\right] \tag{2.1.3}$$

For Launch height $= 0$, the square root in (2.1.3) equals $v_0 \sin(\theta)$. So the bracket equals $2v_0 \sin(\theta)$, and (2.1.3) reduces to (2.1.1).

(2.1.3) also shows how this engine sizes delimiters. Three scalable pairs are nested: the parentheses of the squared group are inside the square root, and the square root is inside the outer bracket. The square root is as tall as a fraction, and the outer bracket must be taller than the square root. Stock KaTeX lets a `\left … \right` pair be slightly shorter than its content. Then all three pairs get the same glyph, and the reader cannot see the nesting. This engine uses a patched KaTeX that makes every pair strictly taller than its content. So each level is visibly larger than the level inside it, and the source contains no manual size command such as `\Bigl` or `\biggl`.

2.1.3 Worked example: try it

The explorer in Figure 2.1.1 evaluates (2.1.1) for a launch speed and a launch angle that you choose. If (2.1.1) is on the screen, a reference to it highlights the equation. If the equation is off the screen, the reference opens a preview of it.

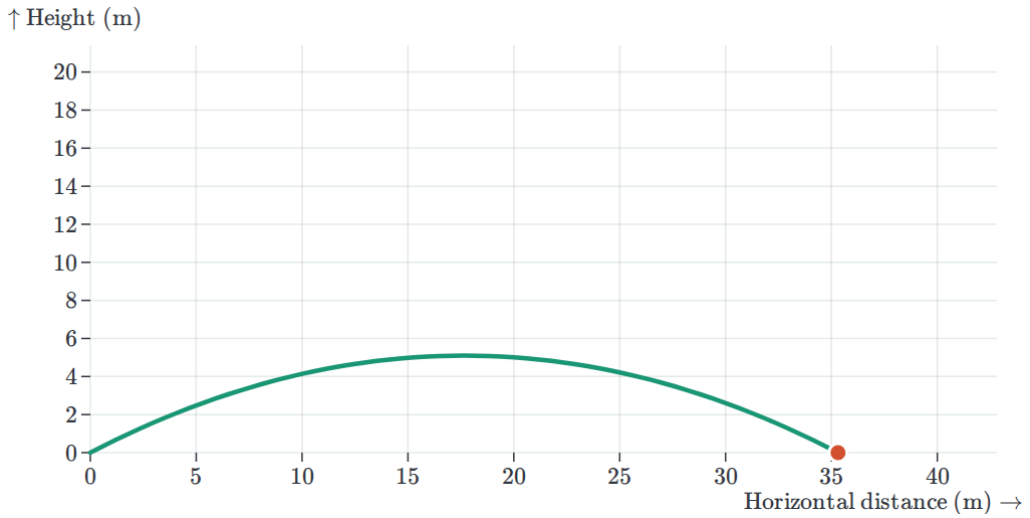


Figure 2.1.1 Projectile trajectory explorer. The path from launch to landing; the dot marks the landing point. Shown for $v_0 = 20$ m/s and $\theta = 30°$: range 35.3 m, time of flight 2.04 s.

For a fixed launch speed, a larger launch angle gives a higher trajectory, but not always a larger range. The range is the horizontal velocity component multiplied by the time of flight. A larger launch angle increases the time of flight but decreases the horizontal velocity component.

The next chapter plots the range against the launch angle for one launch speed, in Figure 2.3.1. That plot shows which launch angle gives the largest range.

A reference to Figure 2.1.1 scrolls back to the explorer.

2.1.4 Sampled flight data

The explorer shows one trajectory at a time. Table 2.1.1 instead lists the state of one flight at eleven values of Time, from launch to landing. The flight has a Launch speed of 45 m/s and a Launch angle of 55°. Each row gives the Position by its Horizontal coordinate and its Vertical coordinate, the Velocity, and the Acceleration from the constant-acceleration equations of the previous part. Each row also gives the Kinetic energy and the Potential energy per unit Mass:

t (s)	x (m)	y (m)	v_x (m/s)	v_y (m/s)	a_x (m/s^2)	a_y (m/s^2)	$\lvert \boldsymbol{v} \rvert$ (m/s)	θ (°)	KE/m (J/kg)	PE/m (J/kg)
0.00	0.00	0.00	25.81	36.86	0.00	-9.80	45.00	55.00	1012.50	0.00
0.75	19.36	24.89	25.81	29.51	0.00	-9.80	39.21	48.83	768.58	243.92
1.50	38.72	44.27	25.81	22.16	0.00	-9.80	34.02	40.65	578.68	433.82
2.25	58.07	58.13	25.81	14.81	0.00	-9.80	29.76	29.85	442.80	569.70
3.00	77.43	66.49	25.81	7.46	0.00	-9.80	26.87	16.12	360.94	651.56
3.76	97.05	69.33	25.81	0.01	0.00	-9.80	25.81	0.03	333.10	679.40
4.50	116.15	66.65	25.81	-7.24	0.00	-9.80	26.81	-15.67	359.30	653.20
5.25	135.51	58.47	25.81	-14.59	0.00	-9.80	29.65	-29.47	439.51	572.99
6.00	154.87	44.77	25.81	-21.94	0.00	-9.80	33.87	-40.36	573.74	438.76
6.75	174.22	25.56	25.81	-29.29	0.00	-9.80	39.04	-48.61	762.00	250.50
7.52	194.17	0.00	25.81	-36.86	0.00	-9.80	45.00	-55.00	1012.50	0.00

Table 2.1.1 Sampled flight data. Position, velocity, acceleration and specific mechanical energy through one flight, from launch to landing.

Eleven columns are enough to make Table 2.1.1 scroll horizontally on most screens. Each row agrees with the results of this chapter:

- The Gravitational acceleration appears only in the a_y column. The a_x column is zero in every row.
- The v_x column has the same value in every row.
- The sum of the last two columns is 1012.50 J/kg in every row. This sum is the mechanical energy per unit Mass. The mechanical energy is constant because air resistance is zero.
- At $t = 3.76$ s, the v_y column is approximately zero. So this row is the highest point of the flight.
- The last row is the landing. For this launch speed and launch angle, (2.1.1) gives $R \approx 194$ m, and the x column gives 194.17 m.

Knowledge check 2.1.1: Projectile range

1. A ball is launched over level ground at 20 m/s, 30° above the horizontal. Using $g = 9.81\ \text{m/s}^2$ and ignoring air resistance, what is its range in meters?

Answers: end of the chapter.

References

- [1] Ling et al., University Physics Volume 1 (2016). ch. 4, "Projectile Motion". https://openstax.org/details/books/university-physics-volume-1
- [2] Urone & Hinrichs, College Physics 2e (2022). ch. 3, "Projectile Motion", Example 3.5. https://openstax.org/details/books/college-physics-2e

Answers to the knowledge checks

Knowledge check 2.1.1: Projectile range

1. **35.3** (accepted within ±0.5). $R = \frac{{v_0}^2 \sin(2\theta)}{g} = \frac{(20\ \text{m/s})^2 \times \sin(60°)}{9.81\ \text{m/s}^2} = \frac{400\ \text{m}^2/\text{s}^2 \times 0.8660}{9.81\ \text{m/s}^2} \approx 35.3\ \text{m}$.

Chapter 2.2 Range Symmetry

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/projectiles/range-symmetry/

The range formula in (2.1.1) predicts a symmetry that is useful when comparing trajectories. With the same Launch speed, a low launch and a high launch can have the same Range.

2.2.1 Complementary angles

Let the Launch angle be θ. Its complementary angle is $90^\circ - \theta$, because the two angles add to 90°. Doubling the complementary angle and using the supplementary-angle identity gives the same sine factor as the original angle:

$$\begin{aligned} \sin(2(90^\circ - \theta)) &= \sin(180^\circ - 2\theta) \\ &= \sin(2\theta) \end{aligned} \tag{2.2.1}$$

The other factors in the range formula do not change. Therefore complementary launch angles give the same range over level ground when air resistance is ignored.

2.2.2 Worked example: matching range

For a launch at 30°, the complementary angle is $90^\circ - 30^\circ = 60^\circ$. Their doubled angles are 60° and 120°, which have the same sine. Therefore launches at 30° and 60° travel the same horizontal distance when their launch speeds are equal.

2.2.3 Different paths, one range

Equal range does not mean equal motion. The steeper launch rises higher and stays in the air longer. The shallower launch has a larger horizontal velocity component, which compensates for its shorter time in the air.

2.2.4 Checking the pairs numerically

A short script computes the range of each launch angle and of its complement (the angle that adds up to 90° with it), for the same launch speed:

range_pairs.py

```
from math import radians, sin

def launch_range(v0, angle_deg, g=9.81):
    return v0**2 * sin(radians(2 * angle_deg)) / g

for angle in (15, 30, 45):
    print(angle, 90 - angle, launch_range(31.6, angle))
```

Its output, one row per pair, is in the table of range pairs. A common slip is to pair 30° with 150° 60°: the two angles add up to 90°, not to 180°. Figure 2.2.1 sums up why each pair shares its range.

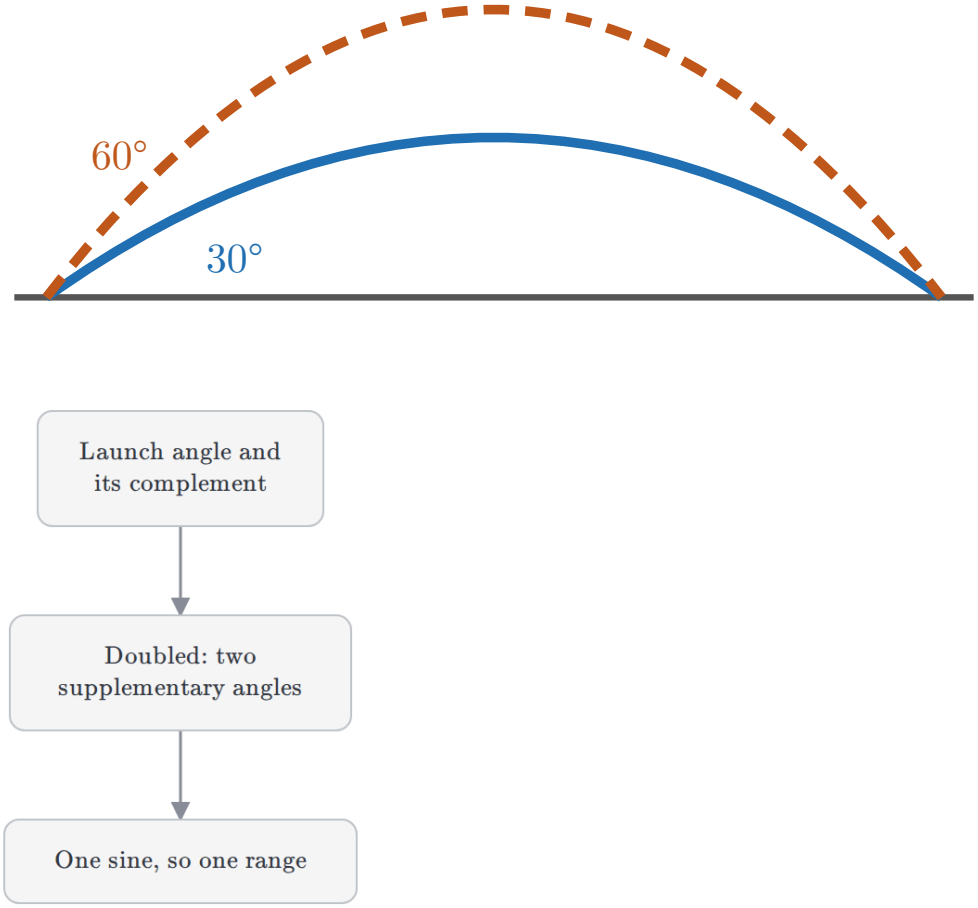


Figure 2.2.1 From an angle to its partner. Each launch angle and its complement double to supplementary angles, which share one sine and so one range.

Before the checks, make sure of both steps:

- ☒ the range formula over level ground
- ☐ the supplementary-angle identity for the sine

Knowledge check 2.2.1: Complementary launch angles

1. A projectile is launched at 35° over level ground with no air resistance. Which other launch angle gives the same range at the same launch speed?
   - (a) 25°
   - (b) 45°
   - (c) 55°

- (d) 65°

Answers: end of the chapter.

Answers to the knowledge checks

Knowledge check 2.2.1: Complementary launch angles

1. **(c)** 55°**.** The angles are complementary: $35^\circ + 55^\circ = 90^\circ$. Their doubled angles are supplementary, so the sine factor in the range formula is equal.

Chapter 2.3 The Optimal Launch Angle

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/projectiles/optimal-launch-angle/

By (2.1.1), the Range depends on the Launch angle only through $\sin(2\theta)$. The previous chapter derives that equation. For $0^\circ \le \theta \le 90^\circ$, the argument 2θ is between 0° and 180°. On this interval, the sine has its maximum value, 1, at 90°. So for a fixed launch speed, the range is largest at $2\theta = 90^\circ$, i.e., at $\theta = 45^\circ$. The explorer in Figure 2.1.1 shows this result. Diagram 2.3.1 shows the inputs to the search for this launch angle.

2.3.1 A polynomial maximum, no calculus

This section finds the same launch angle as the maximum of a polynomial. For $0^\circ \le \theta \le 90^\circ$, the product $\sin(\theta)\cos(\theta)$ is not negative. So the range is largest where the square of this product is largest. Let u be the Squared sine of the launch angle: $u = \sin^2(\theta)$. Rewriting the square of the product in terms of u gives a second-order polynomial:

$$\begin{aligned} \sin^2(\theta)\cos^2(\theta) &= \sin^2(\theta)(1-\sin^2(\theta)) \quad \text{Pythagorean identity} \\ &= u(1-u) \quad \text{definition of } u \\ &= u - u^2 \end{aligned} \tag{2.3.1}$$

The graph of this polynomial is a downward parabola with roots at $u = 0$ and $u = 1$. A downward parabola has its maximum halfway between its roots, so the polynomial is largest at $u = \frac{1}{2}$. Converting this value of u back to a launch angle gives the Optimal launch angle:

$$\begin{aligned} \theta^* &= \arcsin\left(\sqrt{u}\right) \qquad \text{definition of } u \text{, with } \sin(\theta) \ge 0 \\ &= \arcsin\left(\tfrac{1}{\sqrt{2}}\right) \qquad \text{vertex at } u = \tfrac{1}{2} \\ &= 45^\circ \end{aligned} \tag{2.3.2}$$

Setting the derivative of the polynomial to zero gives the same vertex (show how). But the symmetry of the parabola gives the vertex without a derivative. In (2.1.1), the launch speed appears only in the factor ${v_0}^2$, which does not depend on the launch angle. A positive constant factor does not change the position of the maximum. So the Optimal launch angle is 45° for every launch speed. [a] checks the vertex numerically and graphically.

Show the full calculation using calculus.

Take the derivative of the expanded polynomial and set it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}u}(u - u^2) = 1 - 2u = 0 \tag{2.3.3}$$

The solution of this equation is $u = \frac{1}{2}$.

2.3.2 Complementary launch angles give equal ranges

Two launch angles are complementary if their sum is 90°. The Complementary launch angle of θ is $\theta' = 90^\circ - \theta$. On this page the prime marks this angle and does not mean a derivative. For complementary launch angles, $\sin(2\theta') = \sin(180^\circ - 2\theta) = \sin(2\theta)$. So two complementary launch angles give equal ranges. For example, the range at a launch angle of 30° equals the

range at a launch angle of 60°. The Optimal launch angle is the only launch angle that is its own complement.

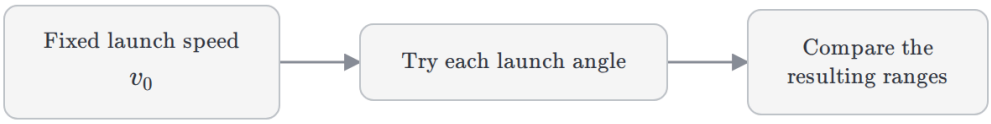


Diagram 2.3.1 Inputs to the launch-angle search. The launch speed stays fixed while each launch angle is tried in turn, and the ranges they give are compared.

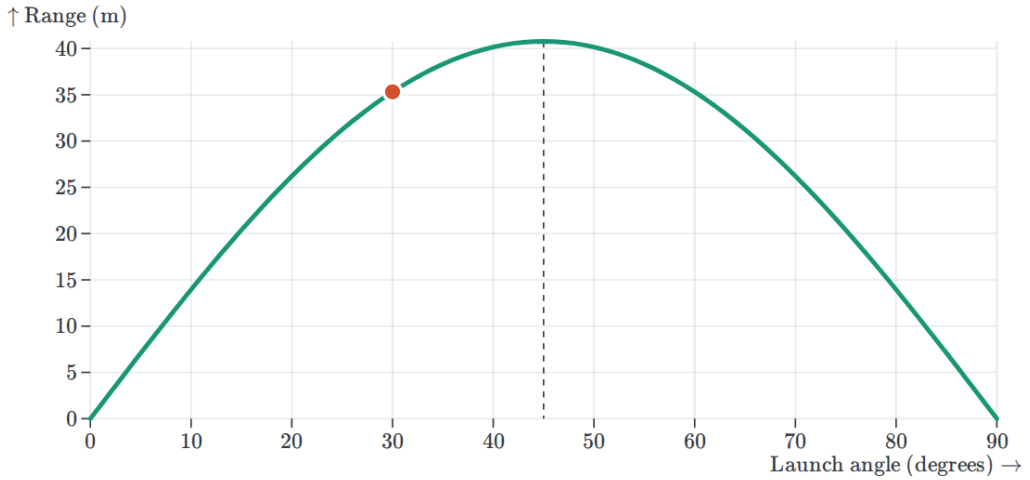


Figure 2.3.1 Range vs. launch angle explorer. The range at every launch angle for one launch speed; the dashed line marks 45°, where the range is largest. Shown for $v_0 = 20$ m/s; the dot marks $\theta = 30°$: range 35.3 m, 5.5 m short of the maximum (40.8 m, at 45°).

The curve is symmetric about 45°. A launch angle of 20° and a launch angle of 70° give the same range. The trajectory at 20° is lower and has a shorter time of flight than the trajectory at 70°.

Memo a: Checking the polynomial vertex: its values on both sides of the vertex, and its downward-parabola shape

Evaluating the polynomial at the vertex, $u = \frac{1}{2}$, gives its maximum value:

$$u(1-u)\big|_{u=1/2} = \tfrac{1}{4} \tag{2.3.4}$$

Two values of u at equal distances from the vertex give equal values of the polynomial, and both values are smaller than the maximum.

Squared sine of the launch angle	Value of the polynomial
0	0
$\frac{1}{4}$	$\frac{3}{16}$
$\frac{1}{2}$	$\frac{1}{4}$
$\frac{3}{4}$	$\frac{3}{16}$
1	0

Table 2.3.1 Polynomial values around the vertex. The polynomial at five inputs placed symmetrically about the vertex, 1/2: inputs equally far from it give equal values.

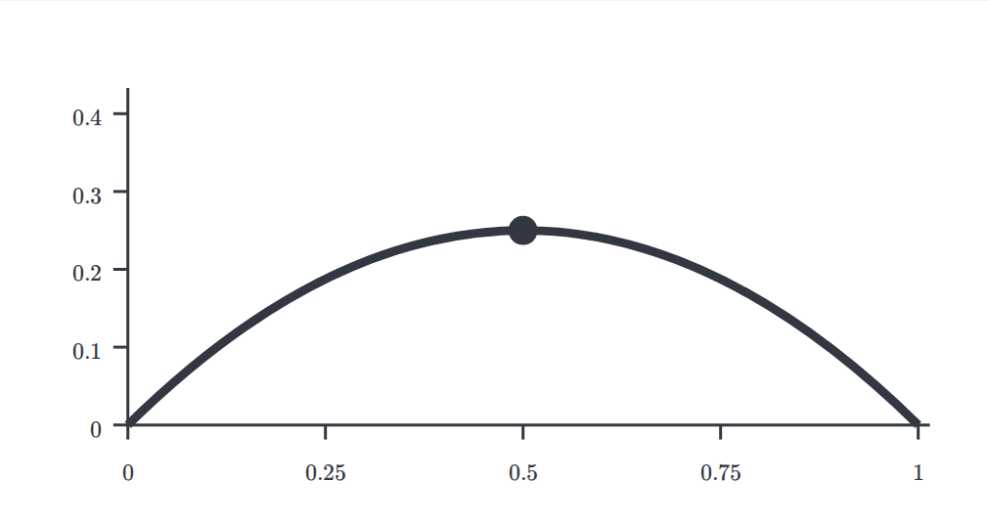


Figure 2.3.2 The polynomial is a downward parabola. From 0 at input 0 it rises to its maximum, 0.25, at input 0.5, and falls symmetrically back to 0 at input 1.

(2.3.4), Table 2.3.1, and Figure 2.3.2 are inside the memo. A reference to each of them also works while the memo is closed.

Knowledge check 2.3.1: Optimal launch angle

1. For a fixed launch speed over level ground, with no air resistance, which launch angle maximizes range?
   - (a) 30°
   - (b) 45°
   - (c) 60°
   - (d) 90°

Answers: end of the chapter.

Answers to the knowledge checks

Knowledge check 2.3.1: Optimal launch angle

1. **(b) 45°.** The range is proportional to $\sin(2\theta)$. For launch angles from 0° to 90°, $\sin(2\theta)$ has its maximum at $2\theta = 90°$. So $\theta^* = 45°$.

Notation

The symbols of this book, in the order they first appear.

On this page, $\square'$ is not an operator. The prime marks the complementary launch angle, not a derivative.

φ **Flight angle**. The angle of the projectile's velocity above the horizontal at one moment of the flight. It equals the launch angle at launch, is zero at the highest point, and is the negative of the launch angle at landing on level ground. *Units:* degrees above the horizontal. *In Chapter 2.1 only.*

a_x **Horizontal acceleration**. The acceleration along the horizontal axis. Without air resistance nothing pushes the projectile sideways, so it is zero for the whole flight. *Units:* meters per second squared. *In Chapter 2.1 only.*

$\square_x$ **Horizontal component**. The part of a vector quantity along the horizontal axis, positive in the direction of flight. *In Chapter 2.1 only.*

KE **Kinetic energy**. The energy of the projectile due to its motion. It equals one half of the mass multiplied by the square of the speed. *Units:* joules. *In Chapter 2.1 only.*

h **Launch height**. The height of the launch point above the plane on which the projectile lands. It is zero for level ground. *Units:* meters. *In Chapter 2.1 only.*

m **Mass**. The projectile's mass. The sampled flight data table gives the kinetic and the potential energy per unit mass, so the mass does not appear in any computed value. *Units:* kilograms. *In Chapter 2.1 only.*

PE **Potential energy**. The energy of the projectile due to its height above the launch point in the gravitational field. *Units:* joules. *In Chapter 2.1 only.*

a_y **Vertical acceleration**. The acceleration along the vertical axis. Gravity alone acts on the projectile, so it equals the negative of the gravitational acceleration for the whole flight. *Units:* meters per second squared. *In Chapter 2.1 only.*

$\square_y$ **Vertical component**. The part of a vector quantity along the vertical axis, positive upward. *In Chapter 2.1 only.*

x **Horizontal coordinate**. The signed distance from the origin along the horizontal axis, measured positive to the right. *Units:* meters.

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

$\boldsymbol{a}$ **Acceleration**. The rate at which velocity changes with time. In this course it is held constant over a scenario, so it is the same vector at every instant. *Units:* meters per second squared.

$$\boldsymbol{a} = \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}t}$$

g **Gravitational acceleration**. The magnitude of gravitational acceleration, treated as constant near the ground. *Units:* meters per second squared.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

$\boldsymbol{r}$ **Position**. The object's location at elapsed time t, taken as a vector from a fixed origin so a two-dimensional motion keeps its horizontal and vertical parts separate. *Units:* meters.

$$\boldsymbol{r} = (x,\, y)$$

R **Range**. Total horizontal distance a projectile covers before it returns to its launch height. *Units:* meters.

$$R = \frac{{v_0}^2 \sin(2\theta)}{g}$$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second.

$$\boldsymbol{v} = \frac{\mathrm{d}\,\boldsymbol{r}}{\mathrm{d}t}\,; \quad v(t) = v_0 + a\,t$$

θ' **Complementary launch angle**. The launch angle whose sum with a given launch angle is 90 degrees. *Units:* degrees above the horizontal. *In Chapter 2.3 only.*

θ^* **Optimal launch angle**. The launch angle that maximizes range for a given fixed launch speed, over level ground. *Units:* degrees above the horizontal. *In Chapter 2.3 only.*

u **Squared sine of the launch angle**. The squared sine of the launch angle, used as the variable in the range polynomial. *Units:* dimensionless. *In Chapter 2.3 only.*

$$u = \sin^2(\theta)$$