Motion in the plane

Course Author

Part 1: Motion in the plane

Chapter 1.1 Position and Coordinate Systems

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/motion/position-and-coordinates/

Every later chapter of this course makes statements about two things: the position of an object, and the change of that position with time. This chapter defines the terms for both. Every quantity is stated in SI units (meters, seconds, kilograms, and joules), so each number has exactly one unit.

1.1.1 Locating a point in the plane

The Position of an object is a vector from a fixed origin to the object. In the plane, the position has two coordinates: the Horizontal coordinate and the Vertical coordinate. Each coordinate is a function of Time. So the position at time t is the pair of the two coordinates at time t:

$$\boldsymbol{r}(t) = (x(t),\, y(t)) \tag{1.1.1}$$

The two coordinates in (1.1.1) are separate functions of the same time t. So the motion of a projectile can be analyzed as two one-dimensional motions, one horizontal and one vertical, with one common time [1].

1.1.2 Two ways to name the same point

The horizontal and the vertical coordinate are the Cartesian coordinates of a point. Polar coordinates are a second way to state the same point. They give the distance of the point from the origin, which is the Polar radius, and the direction of the point from the origin, which is the Polar angle. The two coordinate systems contain the same information, so each converts into the other.

1.1.2.1 From Cartesian to polar

The Pythagorean theorem gives the polar radius from the two Cartesian coordinates, and the arctangent gives the polar angle:

$$\rho = \sqrt{x^2 + y^2}, \qquad \varphi = \arctan\left(\frac{y}{x}\right) \tag{1.1.2}$$

The arctangent in (1.1.2) gives angles between $-90°$ and $90°$ only. These angles cover the right half of the plane. For a point in the left half of the plane, add $180°$ to the result. For example, the points $(-1, -1)$ and $(1, 1)$ have the same ratio y/x, but they are on opposite sides of the origin. The conversion from polar to Cartesian coordinates in § 1.1.2.2 has no such restriction.

1.1.2.2 From polar to Cartesian

The polar radius multiplied by the cosine and by the sine of the polar angle gives the two Cartesian coordinates:

$$x = \rho\cos(\varphi), \qquad y = \rho\sin(\varphi) \tag{1.1.3}$$

A launch velocity is usually stated in polar form: a speed and an angle above the horizontal. The conversion in (1.1.3) applies to every vector in the plane. So it also gives the horizontal and the vertical component of a launch velocity.

1.1.3 Velocity and acceleration as derivatives

The Velocity vector is defined as the time derivative of the position. The derivative of a vector is the vector of the derivatives of its coordinates:

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \tag{1.1.4}$$

A dot over a letter is a short way to write the Time derivative: $\dot{x}$ is the time derivative of the horizontal coordinate. With this notation, (1.1.4) is $\boldsymbol{v}(t) = (\dot{x}, \dot{y})$.

The Acceleration is defined as the time derivative of the velocity. So the acceleration is the second time derivative of the position:

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} \tag{1.1.5}$$

If the position is twice differentiable with respect to time, (1.1.4) and (1.1.5) are correct. Neither equation assumes that the acceleration is constant in size or in direction.

1.1.4 The special case: constant acceleration

Every later scenario in this course adds one assumption: the Acceleration is constant in size and in direction.

1.1.4.1 Integrating once: velocity

A subscript zero marks a quantity's Initial value. With a constant acceleration, integrating (1.1.5) once with respect to time gives the velocity at time t from the initial velocity:

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 + \boldsymbol{a}\,t \tag{1.1.6}$$

1.1.4.2 Integrating twice: position

Integrating (1.1.6) once more with respect to time gives the position at time t from the initial position and the initial velocity:

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0\,t + \tfrac{1}{2}\boldsymbol{a}\,t^2 \tag{1.1.7}$$

The later chapters of this course use (1.1.7) more than any other equation. Its horizontal and its vertical coordinate give two independent equations. For a projectile, gravity accelerates only the vertical motion. So the horizontal velocity component is constant. The vertical velocity component decreases, is zero at the highest point, and is negative after the highest point.

Example 1.1.1: Calculation examples between coordinate systems

Cartesian to polar

A point is at $x = 3$ m and $y = 4$ m. Substituting these coordinates into (1.1.2) gives its Polar radius:

$$\begin{aligned}\rho &= \sqrt{(3\text{ m})^2 + (4\text{ m})^2} \\ &= \sqrt{9\text{ m}^2 + 16\text{ m}^2} \\ &= 5\text{ m}\end{aligned}$$

The point is in the right half of the plane, so its Polar angle is $\arctan(4/3) \approx 53.1^\circ$ above the horizontal (approximately 0.93 rad).

Polar to Cartesian

A ball is thrown with a speed of 10 m/s at 30° above the horizontal. Applying (1.1.3) to this launch velocity gives its horizontal and vertical components:

$$\begin{aligned} \boldsymbol{v}_0 &= (10\ \text{m/s}\cos(30^\circ),\ 10\ \text{m/s}\sin(30^\circ)) \\ &\approx (8.66,\ 5.00)\ \text{m/s} \end{aligned}$$

Point	x (m)	y (m)	ρ (m)	φ (°)
A	3	4	5	53.1
B	8.66	5.00	10	30.0

Table 1.1.1 Two points in both coordinate systems. Points A and B, each with its Cartesian coordinates x and y and its polar coordinates ϱ and φ.

Table 1.1.1 states two points in both coordinate systems. Point A is the point of the first example in Example 1.1.1. Point B has the numbers of the second example, as a position in m instead of a velocity in m/s.

Knowledge check 1.1.1: Position and coordinates

1. A point is at $x = 3$ m and $y = 4$ m. What is its polar radius, in meters?
2. How many times must position be differentiated with respect to time to obtain acceleration?
3. A point has $\rho = 2$ m and $\varphi = 90^\circ$. Where is it?
   - (a) On the positive y axis, 2 m from the origin
   - (b) On the positive x axis, 2 m from the origin
   - (c) On the negative y axis, 2 m from the origin
   - (d) At the origin
4. A point has $\rho = 10$ m and $\varphi = 60^\circ$. What is its coordinate x, in meters?
5. Which polar coordinates describe the point at $x = 0$ and $y = -3$ m?
   - (a) $\rho = 3$ m and $\varphi = 270^\circ$
   - (b) $\rho = 3$ m and $\varphi = 90^\circ$
   - (c) $\rho = 3$ m and $\varphi = 180^\circ$
   - (d) $\rho = 0$ and $\varphi = 270^\circ$

Answers: end of the chapter.

References

- [1] Ling et al., University Physics Volume 1 (2016). ch. 3, “Time, Position, and Displacement”. https://openstax.org/details/books/university-physics-volume-1

Answers to the knowledge checks

Knowledge check 1.1.1: Position and coordinates

1. **5** (accepted within ±0.01). $\rho = \sqrt{x^2 + y^2} = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = \sqrt{9\text{ m}^2 + 16\text{ m}^2} = 5\text{ m}$.
2. **2** (accepted within ±0.01). Velocity is the first time derivative of position. Acceleration is the time derivative of velocity. So acceleration is the second time derivative of position.
3. **(a) On the positive y axis, 2 m from the origin.** $x = \rho\cos(\varphi) = 2\text{ m} \cdot \cos(90^\circ) = 0$ and $y = \rho\sin(\varphi) = 2\text{ m} \cdot \sin(90^\circ) = 2\text{ m}$, so the point is on the positive y axis.
4. **5** (accepted within ±0.01). $x = \rho\cos(\varphi) = 10\text{ m} \cdot \cos(60^\circ) = 10\text{ m} \cdot 0.5 = 5\text{ m}$.
5. **(a) $\rho = 3$ m and $\varphi = 270^\circ$.** $\rho = \sqrt{x^2 + y^2} = \sqrt{0 + (-3\text{ m})^2} = 3\text{ m}$. The point lies on the negative y axis, three quarters of a turn counterclockwise from the positive x axis, so $\varphi = 270^\circ$.

Chapter 1.2 Velocity Under Constant Acceleration

From *Explico Playground — Motion*, with changes after version 1.0.0, not yet released. Interactive version: https://danieltuzes.github.io/explico/playground/motion/constant-acceleration/

The time origin

In every scenario of this course, Time is zero at the start of the scenario. Time only increases during a scenario. An object can reverse its direction of motion, but time does not reverse.

1.2.1 Velocity over time

The Velocity of an object states its speed and its direction of motion at one instant. In this chapter, the object moves along one straight line. So the velocity is a signed number: its absolute value is the speed, and its sign is the direction. The Acceleration is constant during the scenario. On one axis, the vector equation (1.1.6) gives the velocity at time t as the Launch speed plus the acceleration multiplied by t:

$$v(t) = v_0 + at \tag{1.2.1}$$

(1.2.1) is exact, not an approximation. The acceleration is defined as the rate of change of the velocity. If the acceleration is constant, integrating this definition over time gives (1.2.1) without an error term [1]. The same integration for a vector is in § 1.1.4.1.

Example 1.2.1: Speeding up and braking at a constant acceleration

Entering a highway

A car enters a straight on-ramp (the curved roadway that lets vehicles merge onto a highway from a surface street) at a Launch speed of 5 m/s and has a constant Acceleration of 2 m/s^2. Substituting these values and a time of 4 s into (1.2.1) gives its velocity:

$$\begin{aligned} v(4) &= 5\ \text{m/s} + 2\ \text{m/s}^2 \times 4\ \text{s} \\ &= 5\ \text{m/s} + 8\ \text{m/s} \\ &= 13\ \text{m/s} \end{aligned}$$

Braking to a stop

A cyclist moves at a Launch speed of 8 m/s and brakes (an acceleration opposite to the velocity decreases the speed) with a constant Acceleration of -2 m/s^2. Substituting these values and a time of 3 s into (1.2.1) gives its velocity:

$$\begin{aligned} v(3) &= 8\ \text{m/s} + (-2\ \text{m/s}^2) \times 3\ \text{s} \\ &= 8\ \text{m/s} - 6\ \text{m/s} \\ &= 2\ \text{m/s} \end{aligned}$$

Each second of braking takes 2 m/s off the velocity:

Time	Velocity
0 s	8 m/s

Time	Velocity
1 s	6 m/s
2 s	4 m/s
3 s	2 m/s

Point at a symbol, in the text or in an equation, to open its explanation. Click the symbol to keep the explanation open while you read.

Scenario	Launch speed (m/s)	Acceleration (m/s^2)	Time (s)	Velocity (m/s)
Entering a highway	5	2	4	13
Braking to a stop	8	-2	3	2

Table 1.2.1 The two worked examples at a glance. Each scenario's launch speed, acceleration and time, and the velocity they give.

Table 1.2.1 compares the inputs and the results of the two examples in Example 1.2.1. Knowledge check 1.2.1 tests the same calculation.

Knowledge check 1.2.1: Velocity under constant acceleration

1. A car has an initial velocity of 5 m/s and a constant acceleration of 2 m/s^2. What is its velocity, in m/s, after 4 s?

Answers: end of the chapter.

References

- [1] Ling et al., University Physics Volume 1 (2016). ch. 3, “Instantaneous Velocity and Speed”. https://openstax.org/details/books/university-physics-volume-1

Answers to the knowledge checks

Knowledge check 1.2.1: Velocity under constant acceleration

1. **13** (accepted within ±0.1). $v(t) = v_0 + a\,t = 5\ \text{m/s} + 2\ \text{m/s}^2 \times 4\ \text{s} = 5\ \text{m/s} + 8\ \text{m/s} = 13\ \text{m/s}$.

Notation

The symbols of this book, in the order they first appear.

φ **Polar angle**. The angle from the positive horizontal axis to the line joining the origin to the point, measured counterclockwise. *Units:* degrees. *In Chapter 1.1 only.*

ρ **Polar radius**. The straight-line distance from the origin to the point, never negative. *Units:* meters. *In Chapter 1.1 only.*

$\boldsymbol{v}$ **Velocity vector**. The rate of change of position with time, carrying both components at once so the horizontal and vertical motions stay separate. *Units:* meters per second. *In Chapter 1.1 only.*

x **Horizontal coordinate**. The signed distance from the origin along the horizontal axis, measured positive to the right. *Units:* meters.

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

$\boldsymbol{a}$ **Acceleration**. The rate at which velocity changes with time. In this course it is held constant over a scenario, so it is the same vector at every instant. *Units:* meters per second squared.

$$\boldsymbol{a} = \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}t}$$

$\square_0$ **Initial value**. The value of a quantity at the start of the scenario, the instant when time is zero.

$$\square_0 = \square\,(t = 0)$$

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

$\boldsymbol{r}$ **Position**. The object's location at elapsed time t, taken as a vector from a fixed origin so a two-dimensional motion keeps its horizontal and vertical parts separate. *Units:* meters.

$$\boldsymbol{r} = (x,\, y)$$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

$\dot{\square}$ **Time derivative**. The rate of change of a quantity with time, written as a dot over the quantity's letter. *Units:* per second.

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second.

$$\boldsymbol{v} = \frac{\mathrm{d}\,\boldsymbol{r}}{\mathrm{d}t}\,; \quad v(t) = v_0 + a\,t$$