Motion in the plane

Explico Playground — Motion

Course Author

Content with changes after version 1.0.0, not yet released

Where the point is

A position is two coordinates on one clock

$$\mathbf{r}(t) = \Big(x(t),\ y(t)\Big) \tag{1.1.1}$$

- The Horizontal coordinate and the Vertical coordinate are separate functions of Time

$\mathbf{r}$ Position t Time x Horizontal coordinate y Vertical coordinate

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$$\mathbf{r}(t) = \Big(x(t),\ y(t)\Big) \tag{1.1.1}$$

- The Horizontal coordinate and the Vertical coordinate are separate functions of Time
- So a motion in the plane is two straight-line motions that share one time

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Two coordinate systems

Polar coordinates give distance and direction

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$$\rho = \sqrt{x^2 + y^2}, \qquad \varphi = \arctan\left(\frac{y}{x}\right) \tag{1.1.2}$$

ρ Polar radius φ Polar angle

The arctangent sees only the right half-plane

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- The points $(1, 1)$ and $(-1, -1)$ have the same ratio y/x

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- For a point on the left, add 180°

Polar to Cartesian works in every quadrant

$$x = \rho\cos(\varphi), \qquad y = \rho\sin(\varphi) \tag{1.1.3}$$

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- A launch velocity is polar: a speed and an angle above the horizontal
- The same two products give its horizontal and vertical components

The same two points in both systems

Table 1.1.1 Two points in both coordinate systems. Points A and B, each with its Cartesian coordinates x and y and its polar coordinates ρ and φ.

Point	x (m)	y (m)	ρ (m)	φ (°)
A	3	4	5	53.1
B	8.66	5.00	10	30.0

How the point moves

Velocity is the time derivative of position

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \ \frac{dy}{dt} \right) \tag{1.1.4}$$

$\mathbf{v}$ Velocity vector $\dot{}$ Time derivative

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- A dot is the Time derivative: $\mathbf{v}(t) = (\dot{x}, \ \dot{y})$

$\mathbf{v}$ Velocity vector $\dot{}$ Time derivative

Acceleration is the derivative once more

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{1.1.5}$$

- Neither derivative assumes a constant Acceleration

Constant acceleration

One integration gives the velocity

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}\, t \tag{1.1.6}$$

- A subscript zero marks the Initial value value

$\mathbf{v}_0$ Velocity vector evaluated at t $= 0$ $_0$ Initial value

A second integration gives the position

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0\, t + \tfrac{1}{2}\mathbf{a}\, t^2 \tag{1.1.7}$$

$\mathbf{r}_0$ Position evaluated at t = 0

A second integration gives the position

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0\, t + \tfrac{1}{2}\mathbf{a}\, t^2 \tag{1.1.7}$$

- Horizontally, gravity adds nothing: the speed stays constant
- Vertically, the speed falls to zero at the top, then turns negative

$\mathbf{r}_0$ Position evaluated at t = 0

Check

Describe a point in both systems (1/2)

Knowledge check 1.1.1 Position and coordinates

1. A point is at $x = 3$ m and $y = 4$ m. What is its polar radius, in meters?
2. How many times must position be differentiated with respect to time to obtain acceleration?
3. A point has $\rho = 2$ m and $\varphi = 90°$. Where is it?
   (a) On the positive y axis, 2 m from the origin
   (b) On the positive x axis, 2 m from the origin
   (c) On the negative y axis, 2 m from the origin
   (d) At the origin
4. A point has $\rho = 10$ m and $\varphi = 60°$. What is its coordinate x, in meters?

Describe a point in both systems (2/2)

Knowledge check 1.1.1 Position and coordinates

5. Which polar coordinates describe the point at $x = 0$ and $y = -3$ m?
   (a) $\rho = 3$ m and $\varphi = 270°$
   (b) $\rho = 3$ m and $\varphi = 90°$
   (c) $\rho = 3$ m and $\varphi = 180°$
   (d) $\rho = 0$ and $\varphi = 270°$

Velocity over time

On a line, a velocity is a signed number

- Its absolute value is the speed
- Its sign is the direction of motion

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- Its absolute value is the speed
- Its sign is the direction of motion
- Time starts at zero and only increases

A constant acceleration adds the same each second

$$v(t) = v_0 + at \tag{1.2.1}$$

$v(t)$ Velocity v_0 Launch speed

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- The equation is exact: the Acceleration is the rate of change of the velocity

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$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}\,t \tag{1.1.6}$$

$v(t)$ Velocity v_0 Launch speed

Two examples

A car on a ramp gains 8 m/s in 4 s

$$\begin{aligned} v(4) &= 5 \text{ m/s} + 2 \text{ m/s}^2 \times 4 \text{ s} \\ &= 13 \text{ m/s} \end{aligned}$$

Braking takes 2 m/s off every second

$$
\begin{aligned}
v(3) &= 8\ \mathsf{m/s} + \left(-2\ \mathsf{m/s^2}\right) \times 3\ \mathsf{s} \\
&= 2\ \mathsf{m/s}
\end{aligned}
$$

Braking takes 2 m/s off every second

$$\begin{aligned} v(3) &= 8 \text{ m/s} + \left(-2 \text{ m/s}^2\right) \times 3 \text{ s} \\ &= 2 \text{ m/s} \end{aligned}$$

- An acceleration against the velocity lowers the speed

The two examples side by side

Table 1.2.1 The two worked examples at a glance. Each scenario's launch speed, acceleration and time, and the velocity they give.

Scenario	Launch speed (m/s)	Acceleration (m/s^2)	Time (s)	Velocity (m/s)
Entering a highway	5	2	4	13
Braking to a stop	8	-2	3	2

Check

Predict a velocity

Knowledge check 1.2.1 Velocity under constant acceleration

1. A car has an initial velocity of 5 m/s and a constant acceleration of 2 m/s^2. What is its velocity, in m/s, after 4 s?

What you can now do

- Given a point in the plane, convert between Cartesian and polar coordinates, and identify velocity and acceleration as the first and second time derivatives of position.
- Given an initial velocity and a constant acceleration, compute the velocity at a later time.

Notation

$\mathbf{r}$ **Position**. The object's location at elapsed time t, taken as a vector from a fixed origin so a two-dimensional motion keeps its horizontal and vertical parts separate. *Units:* meters. $\mathbf{r} = (x,\ y)$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

x **Horizontal coordinate**. The signed distance from the origin along the horizontal axis, measured positive to the right. *Units:* meters.

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

ρ **Polar radius**. The straight-line distance from the origin to the point, never negative. *Units:* meters.

φ **Polar angle**. The angle from the positive horizontal axis to the line joining the origin to the point, measured counterclockwise. *Units:* degrees.

Notation (continued)

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

$\mathbf{v}$ **Velocity vector**. The rate of change of position with time, carrying both components at once so the horizontal and vertical motions stay separate. *Units:* meters per second.

$\dot{\square}$ **Time derivative**. The rate of change of a quantity with time, written as a dot over the quantity's letter. *Units:* per second.

$\mathbf{a}$ **Acceleration**. The rate at which velocity changes with time. In this course it is held constant over a scenario, so it is the same vector at every instant. *Units:* meters per second squared. $\mathbf{a} = \dfrac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}t}$

Notation (continued)

$\square_0$ **Initial value**. The value of a quantity at the start of the scenario, the instant when time is zero. $\square_0 = \square\,(t = 0)$

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second. $\mathbf{v} = \dfrac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}t}$; $\quad v(t) = v_0 + a\,t$