Motion in the plane

Explico Playground — Motion

Course Author

Content with changes after version 1.0.0, not yet released

Where the point is

A position is two coordinates on one clock

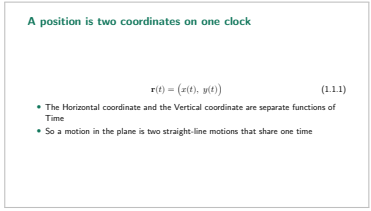
$$\mathbf{r}(t) = \Big(x(t),\ y(t)\Big) \tag{1.1.1}$$

- The Horizontal coordinate and the Vertical coordinate are separate functions of Time
- So a motion in the plane is two straight-line motions that share one time

$\mathbf{r}$ Position t Time x Horizontal coordinate y Vertical coordinate

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A position is two coordinates on one clock



The Position is a vector from a fixed origin to the object. Its two coordinates are separate functions of the same time. Every later chapter uses this split: it solves the horizontal and the vertical motion on their own and joins them through the common time.

Two coordinate systems

Polar coordinates give distance and direction

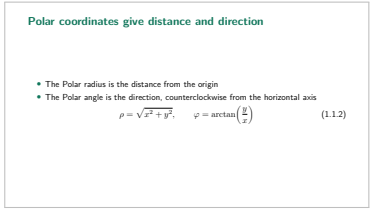
- The Polar radius is the distance from the origin
- The Polar angle is the direction, counterclockwise from the horizontal axis

$$\rho = \sqrt{x^2 + y^2}, \qquad \varphi = \arctan\left(\frac{y}{x}\right) \tag{1.1.2}$$

ρ Polar radius φ Polar angle

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Polar coordinates give distance and direction



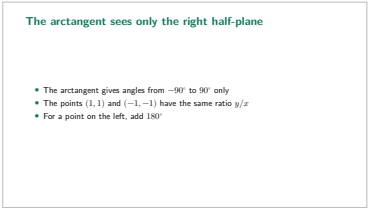
Cartesian and polar coordinates hold the same information, so each converts into the other. The Pythagorean theorem gives the polar radius. The arctangent of the ratio of the two coordinates gives the polar angle.

The arctangent sees only the right half-plane

- The arctangent gives angles from -90° to 90° only
- The points $(1, 1)$ and $(-1, -1)$ have the same ratio y/x
- For a point on the left, add 180°

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The arctangent sees only the right half-plane



Ask the room for the polar angle of the point $(-1, -1)$ before showing the rule. A calculator answers 45°, which is the angle of $(1, 1)$. The two points are on opposite sides of the origin, so their angles differ by 180°.

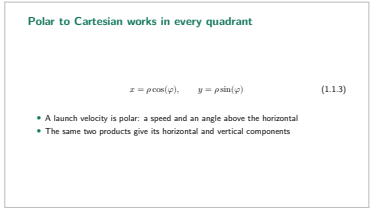
Polar to Cartesian works in every quadrant

$$x = \rho\cos(\varphi), \qquad y = \rho\sin(\varphi) \tag{1.1.3}$$

- A launch velocity is polar: a speed and an angle above the horizontal
- The same two products give its horizontal and vertical components

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Polar to Cartesian works in every quadrant



The cosine and the sine carry the sign of each quadrant, so this direction of the conversion needs no correction. Every later chapter applies it to the launch velocity: the Launch speed times the cosine of the Launch angle is the horizontal component.

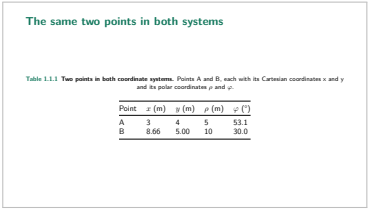
The same two points in both systems

Table 1.1.1 Two points in both coordinate systems. Points A and B, each with its Cartesian coordinates x and y and its polar coordinates ρ and φ.

Point	x (m)	y (m)	ρ (m)	φ (°)
A	3	4	5	53.1
B	8.66	5.00	10	30.0

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The same two points in both systems



The same two points in both systems

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B	8.66	5.00	10	30.0

Point A is the right triangle with sides 3 and 4, so its polar radius is 5 m. Point B has the numbers of a launch at 10 m/s and $30°$, taken as a position instead of a velocity.

How the point moves

Velocity is the time derivative of position

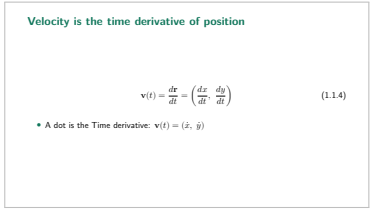
$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \ \frac{dy}{dt} \right) \tag{1.1.4}$$

- A dot is the Time derivative: $\mathbf{v}(t) = (\dot{x}, \ \dot{y})$

$\mathbf{v}$ Velocity vector $\dot{}$ Time derivative

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Velocity is the time derivative of position



The derivative of a vector is the vector of the derivatives of its coordinates. So the Velocity vector keeps the two motions apart, as the position does. The dot over a letter is a short way to write a time derivative, and the course uses it from here on.

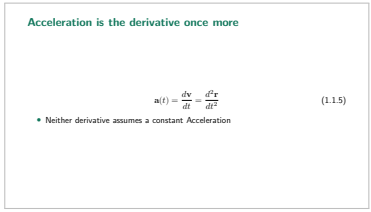
Acceleration is the derivative once more

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{1.1.5}$$

- Neither derivative assumes a constant Acceleration

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Acceleration is the derivative once more



The acceleration is the rate of change of the velocity, so it is the second time derivative of the position. Both definitions hold for any motion whose position is twice differentiable. The next section adds the one assumption that every later chapter makes.

Constant acceleration

One integration gives the velocity

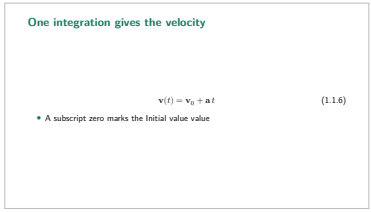
$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}\,t \tag{1.1.6}$$

- A subscript zero marks the Initial value value

$\mathbf{v}_0$ Velocity vector evaluated at t = 0 $_0$ Initial value

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One integration gives the velocity



With a constant acceleration, integrating the acceleration once over time gives the velocity at time t. The constant of the integration is the velocity at time zero.

A second integration gives the position

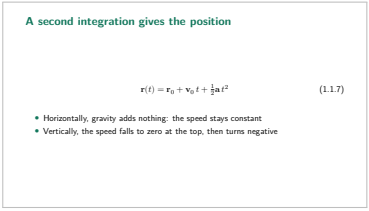
$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0\, t + \tfrac{1}{2}\mathbf{a}\, t^2 \tag{1.1.7}$$

- Horizontally, gravity adds nothing: the speed stays constant
- Vertically, the speed falls to zero at the top, then turns negative

$\mathbf{r}_0$ Position evaluated at t = 0

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A second integration gives the position



The later chapters use this equation more than any other. Its horizontal and its vertical coordinate give two separate equations. For a projectile, gravity accelerates only the vertical motion, so the horizontal velocity component is constant.

Check

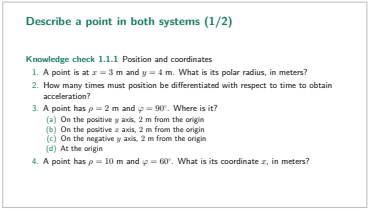
Describe a point in both systems (1/2)

Knowledge check 1.1.1 Position and coordinates

1. A point is at $x = 3$ m and $y = 4$ m. What is its polar radius, in meters?
2. How many times must position be differentiated with respect to time to obtain acceleration?
3. A point has $\rho = 2$ m and $\varphi = 90°$. Where is it?
   (a) On the positive y axis, 2 m from the origin
   (b) On the positive x axis, 2 m from the origin
   (c) On the negative y axis, 2 m from the origin
   (d) At the origin
4. A point has $\rho = 10$ m and $\varphi = 60°$. What is its coordinate x, in meters?

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Describe a point in both systems (1/2)



Answer: 5 (within ± 0.01). $\rho = \sqrt{x^2 + y^2} = \sqrt{(3\ \text{m})^2 + (4\ \text{m})^2} = \sqrt{9\ \text{m}^2 + 16\ \text{m}^2} = 5$ m.
Answer: 2 (within ± 0.01). Velocity is the first time derivative of position. Acceleration is the time derivative of velocity. So acceleration is the second time derivative of position.
Answer: (a) On the positive y axis, 2 m from the origin. $x = \rho\cos(\varphi) = 2\ \text{m} \cdot \cos(90°) = 0$ and $y = \rho\sin(\varphi) = 2\ \text{m} \cdot \sin(90°) = 2$ m, so the point is on the positive y axis.
Answer: 5 (within ± 0.01). $x = \rho\cos(\varphi) = 10\ \text{m} \cdot \cos(60°) = 10\ \text{m} \cdot 0.5 = 5$ m.

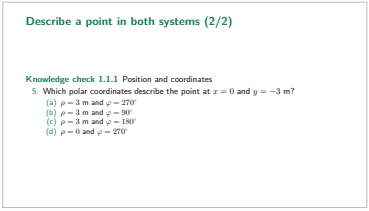
Describe a point in both systems (2/2)

Knowledge check 1.1.1 Position and coordinates

5. Which polar coordinates describe the point at $x = 0$ and $y = -3$ m?
   (a) $\rho = 3$ m and $\varphi = 270^\circ$
   (b) $\rho = 3$ m and $\varphi = 90^\circ$
   (c) $\rho = 3$ m and $\varphi = 180^\circ$
   (d) $\rho = 0$ and $\varphi = 270^\circ$

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Describe a point in both systems (2/2)



Answer: (a) $\rho = 3$ m and $\varphi = 270^\circ$. $\rho = \sqrt{x^2 + y^2} = \sqrt{0 + (-3\text{ m})^2} = 3$ m. The point lies on the negative y axis, three quarters of a turn counterclockwise from the positive x axis, so $\varphi = 270^\circ$.

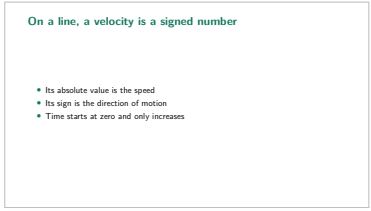
Velocity over time

On a line, a velocity is a signed number

- Its absolute value is the speed
- Its sign is the direction of motion
- Time starts at zero and only increases

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On a line, a velocity is a signed number



In this chapter the object moves along one straight line, so its Velocity is one number with a sign. An object can reverse its direction of motion. Time does not reverse.

A constant acceleration adds the same each second

$$v(t) = v_0 + at \tag{1.2.1}$$

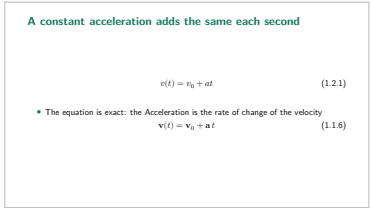
- The equation is exact: the Acceleration is the rate of change of the velocity

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}\,t \tag{1.1.6}$$

$v(t)$ Velocity v_0 Launch speed

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A constant acceleration adds the same each second



Integrating the definition of the acceleration over time gives this equation with no error term, because the acceleration is constant. The velocity at time zero is the Launch speed. It is the first chapter's equation for the velocity vector, read along one line, which is why the second recall carries the first chapter's number.

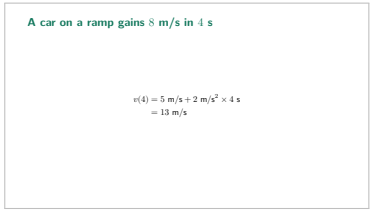
Two examples

A car on a ramp gains 8 m/s in 4 s

$$\begin{aligned} v(4) &= 5 \text{ m/s} + 2 \text{ m/s}^2 \times 4 \text{ s} \\ &= 13 \text{ m/s} \end{aligned}$$

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A car on a ramp gains 8 m/s in 4 s



A car enters a straight on-ramp at 5 m/s with a constant acceleration of 2 m/s^2. In 4 s the acceleration adds 8 m/s, so the car's velocity is 13 m/s.

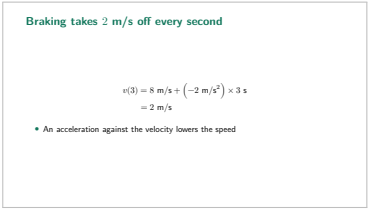
Braking takes 2 m/s off every second

$$\begin{aligned} v(3) &= 8 \text{ m/s} + \left(-2 \text{ m/s}^2\right) \times 3 \text{ s} \\ &= 2 \text{ m/s} \end{aligned}$$

- An acceleration against the velocity lowers the speed

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Braking takes 2 m/s off every second



A cyclist at 8 m/s brakes with a constant acceleration of -2 m/s^2. The acceleration points against the velocity, so the speed falls by 2 m/s every second. Ask the room when the cyclist stops. The answer is after 4 s.

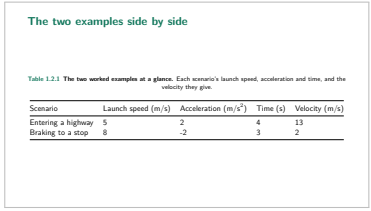
The two examples side by side

Table 1.2.1 The two worked examples at a glance. Each scenario's launch speed, acceleration and time, and the velocity they give.

Scenario	Launch speed (m/s)	Acceleration (m/s^2)	Time (s)	Velocity (m/s)
Entering a highway	5	2	4	13
Braking to a stop	8	-2	3	2

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The two examples side by side



The two examples side by side

Table 1.2.1 **The two worked examples at a glance.** Each scenario's launch speed, acceleration and time, and the velocity they give.

Scenario	Launch speed (m/s)	Acceleration (m/s²)	Time (s)	Velocity (m/s)
Entering a highway	5	2	4	13
Braking to a stop	8	-2	3	2

The table puts the inputs and the result of both examples in one place. The same equation gives both results. Only the sign of the acceleration differs.

Check

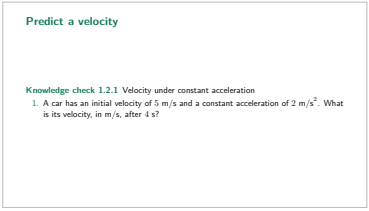
Predict a velocity

Knowledge check 1.2.1 Velocity under constant acceleration

1. A car has an initial velocity of 5 m/s and a constant acceleration of 2 m/s^2. What is its velocity, in m/s, after 4 s?

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Predict a velocity



Answer: 13 (within ± 0.1). $v(t) = v_0 + a\,t = 5\ \text{m/s} + 2\ \text{m/s}^2 \times 4\ \text{s} = 5\ \text{m/s} + 8\ \text{m/s} = 13\ \text{m/s}$.

What you can now do

- Given a point in the plane, convert between Cartesian and polar coordinates, and identify velocity and acceleration as the first and second time derivatives of position.
- Given an initial velocity and a constant acceleration, compute the velocity at a later time.

Notation

$\mathbf{r}$ **Position**. The object's location at elapsed time t, taken as a vector from a fixed origin so a two-dimensional motion keeps its horizontal and vertical parts separate. *Units:* meters. $\mathbf{r} = (x,\ y)$

t **Time**. Elapsed time since the object was launched or released, measured in seconds. *Units:* seconds after launch.

x **Horizontal coordinate**. The signed distance from the origin along the horizontal axis, measured positive to the right. *Units:* meters.

y **Vertical coordinate**. The signed distance from the origin along the vertical axis, measured positive upward. *Units:* meters.

ρ **Polar radius**. The straight-line distance from the origin to the point, never negative. *Units:* meters.

φ **Polar angle**. The angle from the positive horizontal axis to the line joining the origin to the point, measured counterclockwise. *Units:* degrees.

Notation (continued)

v_0 **Launch speed**. The object's speed at the start of the scenario being analyzed — its initial velocity in a straight-line problem, or its launch speed the instant a projectile leaves the ground. *Units:* meters per second.

θ **Launch angle**. The angle between a projectile's initial velocity and the horizontal ground at launch. *Units:* degrees above the horizontal.

$\mathbf{v}$ **Velocity vector**. The rate of change of position with time, carrying both components at once so the horizontal and vertical motions stay separate. *Units:* meters per second.

$\dot{\square}$ **Time derivative**. The rate of change of a quantity with time, written as a dot over the quantity's letter. *Units:* per second.

$\mathbf{a}$ **Acceleration**. The rate at which velocity changes with time. In this course it is held constant over a scenario, so it is the same vector at every instant. *Units:* meters per second squared. $\mathbf{a} = \dfrac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}t}$

Notation (continued)

$\square_0$ **Initial value**. The value of a quantity at the start of the scenario, the instant when time is zero. $\square_0 = \square\,(t = 0)$

$v(t)$ **Velocity**. The rate at which position changes with time. It is a vector; in a straight-line problem only its size and sign matter, and that is what the scalar v(t) tracks. *Units:* meters per second. $\mathbf{v} = \dfrac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}t}$; $\quad v(t) = v_0 + a\,t$