Derivative foundations

Part 4 of Credit Products Playground

danieltuzes

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https://danieltuzes.github.io/credit-and-derivatives/

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Part 4

Derivative foundations

Part 1 gave present value, no-arbitrage replication, and risk-neutral pricing as a discounted expectation over a finite set of states. This part applies those results to the first derivative contracts. Replication fixes the forward delivery price and put-call parity, and the risk-neutral expectation values an option, first on a one-period tree and then node by node on a multi-period lattice.

This part covers forwards and European options on one underlying. The first chapter separates a forward's obligation, its fair delivery price, and the value of an existing contract. The second chapter separates an option's right and expiry payoff from its current value, premium, and profit. The third chapter matches call and put portfolios to derive put-call parity. The last two chapters value an option: on a one-period binomial tree by replication and by risk-neutral weights, and on a multi-period lattice by backward induction with local discount factors.

The argument

1. A forward is an obligation, and its delivery price makes a new contract worth zero
2. An option is a right, and its payoff at expiry is not its value today
3. Matched call and put portfolios give put-call parity
4. A two-state payoff is replicated, or weighted by risk-neutral probabilities
5. Backward induction repeats the one-period step node by node

After this part you can

- Interpret the long and short obligations, delivery time, delivery price, and signed expiry payoff of a forward contract.
- Calculate the fair delivery price of a newly struck forward from spot value, present value of deterministic income before delivery, and the delivery-date discount factor.
- Calculate the signed value to the long of an existing forward from the current forward price, its fixed delivery price, quantity, and the delivery-date discount factor.
- Identify the underlying, call or put right, holder and writer, strike, expiry, and settlement perspective of a European option.
- Calculate the expiry payoff of a long or short European call or put from the underlying expiry value, strike, and quantity.
- Distinguish an option's current value or premium from its expiry payoff and from profit after accounting for the premium under a stated financing convention.
- Explain European put-call parity by matching the expiry cash flows of a call-minus-put portfolio and a forward-style position under stated income and discounting assumptions.
- Calculate a missing European call or put value from the other option value, prepaid forward value, strike, and expiry discount factor.
- Calculate the underlying units and risk-free cash position that replicate a claim's two one-period state payoffs.

- Calculate a European option's one-period binomial value using model-implied risk-neutral weights and deterministic discounting.
- Calculate node values on an input multi-period recombining pricing lattice by risk-neutral expectation and one-period node discounting.

Builds on

- Foundations

Leads to

- Bonds
- Bond options

Chapter 4.1

Forward contracts, delivery price, and value

Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/forward-contracts-and-

4.1.1 What you will be able to do

After this lesson, you should be able to:

- read the long and short obligations of a forward;
- distinguish delivery price, current forward price, and contract value;
- calculate a fair delivery price with deterministic income;
- calculate the signed value of an existing forward.

4.1.2 An obligation, not an option

The exchange occurs at the forward-delivery-time T_{fwd}. Under the physical-settlement convention, the long pays the forward-delivery-price K_{fwd} and receives one unit of the asset. If the derivative-underlying-value at delivery is $S_{T_{\text{fwd}}}$, the local long-forward-expiry-payoff per unit is the asset value minus the delivery price:

$$\Pi^{\text{long}}_{T_{\text{fwd}}} = S_{T_{\text{fwd}}} - K_{\text{fwd}} \tag{4.1.1}$$

The payoff of the short is the negative of the payoff of the long. Hull distinguishes this bilateral obligation from an option holder's exercise right and gives the corresponding long and short payoff directions[1].

4.1.3 Fair delivery price at inception

The forward-price $F_{0,T_{\text{fwd}}}$ is the delivery price that would make a new contract worth zero at valuation-time 0. Suppose the underlying is worth S_0 and the underlying-income-present-value paid before delivery is I_0. The long does not own the asset before delivery and does not receive the income. So the prepaid value of the asset delivered is $S_0 - I_0$.

Under the stated cash-and-carry assumptions, with the delivery-time discount-factor $D(0, T_{\text{fwd}})$, the law of one price gives the forward price:

$$F_{0,T_{\text{fwd}}} = \frac{S_0 - I_0}{D(0, T_{\text{fwd}})} \tag{4.1.2}$$

Hull derives the no-income and known-income forward prices by matching a spot purchase and financing strategy to the forward exchange [2].

For $S_0 = 100$, $I_0 = 4$, and $D(0, T_{\text{fwd}}) = 0.95$, the forward price is:

$$F_{0,T_{\text{fwd}}} = \frac{100 - 4}{0.95} = 101.0526316 \tag{4.1.3}$$

The delivery price is not a premium paid at inception. A new contract with a fair delivery price has signed value zero, because its delivery price equals the current forward price.

4.1.4 An existing forward can gain or lose value

After inception, K_{fwd} stays fixed, but the current forward price can change. For the local forward-contract-quantity N, the forward-contract-value to the long at time zero is the discounted difference between the current forward price and the delivery price:

$$V_0^{\text{fwd}} = N D(0, T_{\text{fwd}}) \left(F_{0,T_{\text{fwd}}} - K_{\text{fwd}} \right) \tag{4.1.4}$$

If $N = 10$, the current forward price is 98, the contract delivery price is 102, and the discount factor is 0.94, the value to the long is:

$$V_0^{\text{fwd}} = 10(0.94)(98 - 102) = -37.60 \tag{4.1.5}$$

The short's value is USD +37.60 under the same inputs. Hull derives this discounted forward-price difference for an existing contract [3].

Knowledge check 4.1.1: Forward contracts and value

1. What does the long side of a physically settled forward contract agree to do at delivery?
   (a) Buy the underlying for the fixed delivery price
   (b) Choose whether to buy, with no obligation
   (c) Receive the delivery price as an inception premium
2. A long forward has delivery price USD 80 and the underlying is worth USD 74 at delivery. What is the long's payoff per unit?
   (a) USD -6
   (b) USD +6
   (c) USD 0 because the long can decline delivery
3. An investment asset is worth USD 100, pays no income before delivery, and D(0,T) = 0.95. What fair delivery price gives a new forward zero value, in USD per unit?
4. A bond's current full price is USD 900. A known coupon before delivery has present value USD 40, and D(0,T) = 0.96. What is the fair full forward delivery price in USD?

5. The current forward price is USD 110, an existing long forward's delivery price is USD 105, and D(0,T) = 0.97. What is the contract value to the long per unit, in USD?

6. A long forward covers 10 units. The current forward price is USD 98 per unit, its fixed delivery price is USD 102, and D(0,T) = 0.94. What is its signed value to the long in USD?

Answers: *end of the chapter.*

4.1.5 Model boundary and review note

This lesson uses a toy cash-and-carry model for an investment asset with known income and deterministic discounting. It does not model futures-style daily settlement, repo specialness, collateral, counterparty credit, taxes, borrowing asymmetry, delivery options, or an asset that cannot be financed or shorted on the assumed terms.

The formulas are implemented in a pure tested domain module. Sources, notation, examples, code, and answer keys remain `draft` pending independent human review.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 1 §1.3, printed pp. 5-7, forward contracts and payoffs.

[2] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 5 §§5.3-5.5, printed pp. 103-108, cash-and-carry forward pricing and known income.

[3] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 5 §5.7, printed pp. 109-111, valuation of existing forward contracts.

Answers to the knowledge checks

Knowledge check 4.1.1: Forward contracts and value

1. **(a)** Buy the underlying for the fixed delivery price. A forward is a bilateral obligation. At delivery the long pays the fixed delivery price and receives the underlying under the stated physical-settlement convention.

2. **(a)** USD -6. The long payoff is underlying value minus delivery price: 74 - 80 = USD -6. Unlike an option holder, the forward long cannot simply decline delivery in this model.

3. 105.2631578947 (accepted within ±0.000001). The prepaid forward is USD 100. Divide by the delivery-date discount factor: 100 / 0.95 = USD 105.2631578947.

4. 895.8333333333 (accepted within ±0.000001). Subtract income received before delivery, then carry the prepaid value to delivery: (900 - 40) / 0.96 = USD 895.8333333333.

5. 4.85 (accepted within ±0.000001). Discount the delivery-date advantage: 0.97 × (110 - 105) = USD 4.85.

6. −37.6 (accepted within ±0.000001). The signed value is 10 × 0.94 × (98 - 102) = USD -37.60.

Chapter 4.2

European option contracts and payoffs

Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/european-option-contra

4.2.1 What you will be able to do

After this lesson, you should be able to:

- identify the holder, writer, underlying, strike, and expiry of a European option;
- calculate long and short call and put payoffs;
- distinguish current value, premium, expiry payoff, and profit.

4.2.2 Rights and obligations

A European call gives its holder the right, but not the obligation, to buy one unit of the underlying for the option-strike-price K at the option-expiry-time T_{opt}. A European put gives the holder the corresponding right to sell. If the holder exercises, the writer must carry out the other side of the exchange. Hull introduces call and put rights and European exercise [1].

4.2.3 Expiry payoff

Let the derivative-underlying-value at expiry be $S_{T_{\text{opt}}}$. The local long-call-expiry-payoff and long-put-expiry-payoff, per unit, are

$$\Pi^{\text{call}}_{T_{\text{opt}}} = \max\Big(S_{T_{\text{opt}}} - K, 0\Big) \tag{4.2.1}$$

$$\Pi^{\text{put}}_{T_{\text{opt}}} = \max\Big(K - S_{T_{\text{opt}}}, 0\Big) \tag{4.2.2}$$

The maximum with zero expresses the holder's right not to exercise when exercise would give a negative payoff. A writer's signed payoff is the negative of the matching holder payoff.

For $S_{T_{\text{opt}}} = 115$ and $K = 100$, one call pays 15 and one put pays zero. For $S_{T_{\text{opt}}} = 70$ and $K = 85$, one put pays 15. A writer of three such puts has a signed payoff of -45.

4.2.4 Value, premium, payoff, and profit are different

The European call value c_t or European put value p_t is a current value. At inception, the holder pays the local option-premium-paid π_0, under the stated settlement convention. The payoff is determined later, by the state at expiry.

If financing is ignored, the local option-holder-expiry-profit of a call is the payoff minus the premium:

$$G_{T_{\text{opt}}} = \Pi^{\text{call}}_{T_{\text{opt}}} - \pi_0 \tag{4.2.3}$$

For example, a call bought for 6 that later pays 9 has a payoff of 9 and a profit of 3, if financing is ignored. A profit calculation that includes financing first moves the premium and all other cash flows to one common date. Hull's option examples distinguish premium, exercise, payoff, and profit [2].

Knowledge check 4.2.1: European option contracts and payoffs

1. What right does the holder of a European call have in this lesson?
   (a) The right to buy the underlying at the strike at expiry
   (b) The right to sell the underlying at any time before expiry
   (c) An obligation to buy the underlying at expiry
2. Who has the exercise right and who must perform if a European put is exercised?
   (a) The holder chooses; the writer must perform
   (b) The writer chooses; the holder must perform
   (c) Both sides independently choose whether to perform
3. Two European calls each have strike USD 100. The underlying is worth USD 115 at expiry. What is the total holder payoff in USD?
4. A trader writes three European puts with strike USD 85. The underlying is worth USD 70 at expiry. What is the writer's signed payoff in USD?
5. A call is purchased today for a USD 6 premium and later pays USD 9 at expiry. Ignoring financing, which quantities are correctly labeled?
   (a) Expiry payoff USD 9; profit USD 3
   (b) Expiry payoff USD 3; profit USD 9
   (c) Both payoff and profit are USD 9
6. An out-of-the-money option expires with zero payoff. Does that imply its value was zero on every earlier date?
   (a) No, earlier value reflected the remaining possible future payoffs
   (b) Yes, a zero realized payoff makes every earlier value zero
   (c) Yes, option value always equals expiry intrinsic value

Answers: end of the chapter.

4.2.5 Model boundary and review note

This lesson covers European calls and puts only. It does not value them or cover American or Bermudan exercise, automatic exercise, assignment, settlement delay, corporate actions, collateral, counterparty default, or barrier and other path-dependent terms.

The payoff functions are implemented in a pure tested domain module. Sources, notation, examples, code, and answer keys remain `draft` pending independent human review.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 1 §1.5, printed pp. 7-9, call and put option rights and European exercise.

[2] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 9 practice questions 9.1-9.2 and 9.9-9.10, printed pp. 211-212, payoff versus profit.

Answers to the knowledge checks

Knowledge check 4.2.1: European option contracts and payoffs

1. **(a)** The right to buy the underlying at the strike at expiry. A European call holder has a right, not an obligation, to buy at the strike at the single expiry time.

2. **(a)** The holder chooses; the writer must perform. The option premium buys the holder the exercise right; the writer bears the corresponding obligation under the stated contract.

3. 30 (accepted within ± 0.000001). Each call pays max(115 - 100, 0) = USD 15. Two calls pay USD 30.

4. -45 (accepted within ± 0.000001). Each holder receives max(85 - 70, 0) = USD 15, so the writer pays 3×15 = USD 45 and has payoff USD -45.

5. **(a)** Expiry payoff USD 9; profit USD 3. The contract payoff is USD 9. Ignoring financing, profit subtracts the USD 6 premium and is USD 3.

6. **(a)** No, earlier value reflected the remaining possible future payoffs. Before expiry, more than one future state may remain possible. A realized zero payoff does not retrospectively erase the value of the contingent right.

Chapter 4.3

Put-call parity and synthetic forwards

Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/put-call-parity/

4.3.1 What you will be able to do

After this lesson, you should be able to:

- show that long call plus short put has a forward-style expiry payoff;
- state the exact contract terms that must match;
- solve a missing European call or put value using discounted strike and known income.

4.3.2 Match the expiry cash flows

For the same derivative-underlying-value, option-strike-price K, and option-expiry-time T_{opt}, the holder payoffs satisfy this identity in every state at expiry:

$$\max\Big(S_{T_{\text{opt}}} - K, 0\Big) - \max\Big(K - S_{T_{\text{opt}}}, 0\Big) = S_{T_{\text{opt}}} - K \tag{4.3.1}$$

The left side is long call plus short put. The right side is the payoff of a long forward-style exchange at strike K. This is an identity in every expiry state, not a forecast or an average.

4.3.3 Present-value parity

Let the underlying-income-present-value be I_0. The local prepaid-forward-value is $\widetilde{F}_{0,T_{\text{opt}}} = S_0 - I_0$. Applying the law of one price to the matched portfolios, with European call value c_0, European put value p_0, and the strike-payment discount-factor $D(0, T_{\text{opt}})$, gives put-call parity in present-value form:

$$c_0 - p_0 = \widetilde{F}_{0,T_{\text{opt}}} - KD(0, T_{\text{opt}}) \tag{4.3.2}$$

The prepaid forward value is the discounted forward-price $F_{0,T_{\text{opt}}}$ for delivery at option expiry:

$$\widetilde{F}_{0,T_{\text{opt}}} = D(0, T_{\text{opt}})F_{0,T_{\text{opt}}} \tag{4.3.3}$$

Here $F_{0,T_{\text{opt}}}$ is the current forward price, not the fixed delivery price of an existing forward. Substituting it into the present-value form gives put-call parity in forward form:

$$c_0 - p_0 = D(0, T_{\text{opt}}) \left(F_{0,T_{\text{opt}}} - K \right) \tag{4.3.4}$$

Hull derives put-call parity by comparing the expiry values of two portfolios and invoking no arbitrage[1].

For $c_0 = 8$, $\widetilde{F}_{0,T_{\text{opt}}} = 97$, $K = 100$, and $D(0, T_{\text{opt}}) = 0.95$, the present-value form gives the put value:

$$p_0 = 8 - 97 + 100(0.95) = 6 \tag{4.3.5}$$

4.3.4 When put-call parity does not apply

The equality needs the same underlying, strike, expiry, exercise style, quantity, income treatment, and settlement basis. American exercise adds a choice of exercise time. A knockout triggered by default changes the cash flows in the default state. If both options terminate on issuer default, the call payoff minus the put payoff is zero in that state. So the matching forward-style claim must also terminate on issuer default. The formula for options without a knockout does not apply unchanged.

Knowledge check 4.3.1: Put-call parity

1. Which matching conditions are required for the European call and put in the lesson's put-call parity?
   (a) Same underlying, strike, expiry, exercise style, and settlement basis
   (b) Only the same current premium
   (c) Only the same call-or-put direction
2. Does vanilla European put-call parity automatically apply to an option that is extinguished by issuer default?
   (a) No, the default state changes the matched cash flows and needs an explicit survival-contingent parity
   (b) Yes, the same formula applies to every contract called an option
   (c) Yes, because default can be ignored after a strike is fixed
3. A European call is worth USD 8, the prepaid forward value is USD 97, the strike is USD 100, and D(0,T) = 0.95. What put value satisfies parity, in USD?
4. A European put is worth USD 7, the prepaid forward value after known income is USD 92, the strike is USD 100, and D(0,T) = 0.96. What call value satisfies parity, in USD?

Answers: end of the chapter.

4.3.5 Model boundary and review note

Put-call parity does not depend on an option pricing model. It still relies on the stated assumptions about trading, funding, income, and matching contract terms. Put-call parity does not give an

option price by itself, and it contains no volatility model.

The parity functions are implemented in a pure tested domain module. Sources, notation, examples, code, and answer keys remain `draft` pending independent human review.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 10 §10.4, printed pp. 221-225, European put-call parity and arbitrage portfolios.

Answers to the knowledge checks

Knowledge check 4.3.1: Put-call parity

1. **(a)** Same underlying, strike, expiry, exercise style, and settlement basis. Parity compares portfolios whose state-by-state expiry cash flows match. Different underlying, strike, expiry, style, or settlement basis breaks that match.

2. **(a)** No, the default state changes the matched cash flows and needs an explicit survival-contingent parity. If both options terminate on default, call-minus-put is also zero in that state. The corresponding synthetic forward must carry the same default contingency.

3. 6 (accepted within ± 0.000001). From c - p = prepaid forward - K D(0,T), p = 8 - 97 + 100 $\times$ 0.95 = USD 6.

4. 3 (accepted within ± 0.000001). Rearrange parity: c = p + prepaid forward - K D(0,T) = 7 + 92 - 96 = USD 3.

Chapter 4.4

One-period binomial option valuation

Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/one-period-binomial-op

4.4.1 What you will be able to do

After this lesson, you should be able to:

- construct the two-position portfolio that matches both state payoffs;
- calculate the tree's risk-neutral up-state weight;
- show that replication and discounted risk-neutral expectation give the same value.

4.4.2 Two states, two traded positions

Let the underlying be worth S_0 at valuation-time and take exactly the local values binomial-up-underlying-value S^{up} or binomial-down-underlying-value S^{down} at the option-expiry-time T_{opt}. A claim pays local binomial-up-claim-payoff X^{up} or binomial-down-claim-payoff X^{down}.

The local binomial-hedge-units Δ is the number of units of the underlying that matches the difference between the two claim payoffs:

$$\Delta = \frac{X^{\text{up}} - X^{\text{down}}}{S^{\text{up}} - S^{\text{down}}} \tag{4.4.1}$$

The local binomial-risk-free-expiry-cash is then chosen so that the portfolio matches the claim payoff in the down state:

$$M_{T_{\text{opt}}} = X^{\text{down}} - \Delta\, S^{\text{down}} \tag{4.4.2}$$

So the local binomial-claim-value is the value of the replicating portfolio at time zero:

$$V_0 = \Delta\, S_0 + D(0, T_{\text{opt}}) M_{T_{\text{opt}}} \tag{4.4.3}$$

4.4.3 The same value with risk-neutral weights

The local binomial-risk-neutral-up-weight q is chosen so that the model reproduces the current value of the underlying:

$$q = \frac{S_0/D(0,T_{\text{opt}}) - S^{\text{down}}}{S^{\text{up}} - S^{\text{down}}} \tag{4.4.4}$$

Under the risk-neutral-probability-measure $\mathbb{Q}$, the claim value is the discounted risk-neutral expected payoff:

$$V_0 = D(0,T_{\text{opt}}) \left(q\, X^{\text{up}} + (1-q)\, X^{\text{down}} \right) \tag{4.4.5}$$

Hull derives the one-step hedge and then shows that its no-arbitrage value equals the discounted risk-neutral expected payoff [1].

4.4.4 Worked call example

Let $S_0 = 100$, $S^{\text{up}} = 120$, $S^{\text{down}} = 80$, $D(0,T_{\text{opt}}) = 1/1.05$, and let the option-strike-price be $K = 100$. The call payoffs are $X^{\text{up}} = 20$ and $X^{\text{down}} = 0$.

Then $\Delta = 0.5$, $M_{T_{\text{opt}}} = -40$, and $q = 0.625$. Both methods give the same claim value:

$$V_0 = 0.5(100) + \frac{-40}{1.05} = \frac{0.625(20) + 0.375(0)}{1.05} = 11.9047619 \tag{4.4.6}$$

The negative cash position is borrowing. It is not a negative option value.

4.4.5 From one period to many periods

Repeating this valuation one period at a time, from expiry backward, is backward induction on a multi-period tree. The later bond-option lessons need a discount factor at each node and an additional state for survival and default. An option with a knockout triggered by default cannot be valued from the distribution at expiry alone, because the option can terminate before expiry.

Knowledge check 4.4.1: One-period binomial option valuation

1. A no-income underlying is worth USD 100 now and will be USD 120 or USD 80 at expiry. D(0,T) = 1/1.05. A call with strike USD 100 pays USD 20 or USD 0. The model-implied Q up-weight is 0.625. What is the call value in USD?

2. In the same tree, a put with strike USD 100 pays USD 0 in the up state and USD 20 in the down state. The Q up-weight is 0.625 and D(0,T) = 1/1.05. What is the put value in USD?

3. A claim pays USD 20 when the underlying is USD 120 and USD 0 when it is USD 80. How many underlying units are required in its one-period replicating portfolio?

4. A claim pays USD 10 when the underlying is USD 130 and USD -2 when it is USD 90. Its hedge quantity is 0.3 underlying units. What risk-free cash amount is required at expiry to reproduce the down-state payoff, in USD?

Answers: *end of the chapter.*

4.4.6 Model boundary and review note

This complete two-state model has one non-income-paying traded underlying, one risk-free account, and deterministic discounting. It does not estimate real-world probabilities or volatility. It does not model multiple time steps, stochastic rates, default, or transaction and funding effects.

The two valuation methods are implemented together in a pure tested domain module. The hand-calculated answer and implementation both remain subject to independent human quantitative review.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 12 §§12.1-12.2, printed pp. 253-259, one-step binomial replication and risk-neutral valuation.

Answers to the knowledge checks

Knowledge check 4.4.1: One-period binomial option valuation

1. 11.9047619048 (accepted within ±0.000001). Discount the Q-weighted payoff: $(1/1.05) \times [0.625 \times 20 + 0.375 \times 0] =$ USD 11.9047619048.

2. 7.1428571429 (accepted within ±0.000001). The Q down-weight is 0.375. Discounting gives $(1/1.05) \times [0.625 \times 0 + 0.375 \times 20] =$ USD 7.1428571429.

3. 0.5 (accepted within ±0.000001). The hedge quantity is the payoff difference divided by the underlying-value difference: (20 - 0) / (120 - 80) = 0.5.

4. −29 (accepted within ±0.000001). Cash at expiry is $M_{T_{\text{opt}}} = X^{\text{down}} - \Delta\, S^{\text{down}} = -2 - 0.3 \times 90 =$ USD −29. A negative amount is borrowing.

Chapter 4.5

Multi-period lattice valuation

Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/multiperiod-lattice-va

4.5.1 Repeat the one-period argument backward

The one-period binomial lesson valued two terminal payoffs. A multi-period lattice applies the same pricing step at every node, beginning with the last row and moving backward to valuation-time.

At the node with lattice-time-index i and lattice-state-index j, let the lattice-node-up-weight be $q_{i,j}$ and the lattice-node-discount-factor be $d_{i,j}$. Let the local next-date cash flow $CF_{i+1,j}$ be paid at the successor node $(i+1, j)$; the node value $V_{i+1,j}$ excludes it. The lattice-node-value at node (i, j) is the discounted risk-neutral expectation of the successor cash flows plus the successor node values:

$$V_{i,j} = d_{i,j}\left[\left(1 - q_{i,j}\right)\left(CF_{i+1,j} + V_{i+1,j}\right) + q_{i,j}\left(CF_{i+1,j+1} + V_{i+1,j+1}\right)\right] \tag{4.5.1}$$

The two successors of node (i, j) are the nodes $(i+1, j)$ and $(i+1, j+1)$. The labels "up" and "down" are conventions. The input up weights and values must use the same order.

4.5.2 Timing is part of the model

In (4.5.1), each next-date cash flow is added to the value of its successor node before discounting. So the value of a coupon bond at a node is ex-coupon: it excludes the coupon paid at that node. A change of the order of exercise and coupon payment changes the contract, and it can change the option payoff.

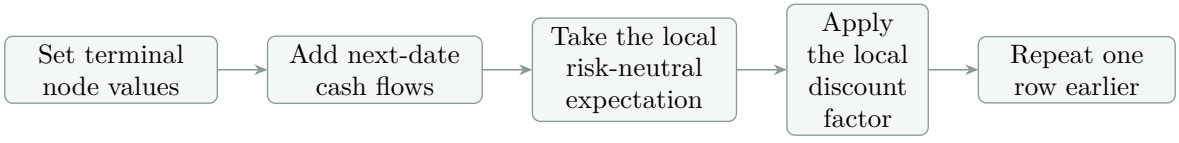


Diagram 4.5.1 **Backward-induction order.** Start from terminal node values. At each earlier row, add any next-date cash flow to each successor value, form the local risk-neutral weighted average, multiply by the local discount factor, and repeat.

4.5.3 Worked example

The terminal values are USD 0, USD 20, and USD 40. At time index 1, both nodes have discount factor 0.90, and their up weights are 0.25 and 0.75. At time index 0, the discount factor is 0.95 and the up weight is 0.50. Applying (4.5.1) row by row gives the node values:

$$\begin{aligned} V_{1,0} &= 0.90\left[(1-0.25)\,(0)+0.25\,(20)\right] = 4.50 \\ V_{1,1} &= 0.90\left[(1-0.75)\,(20)+0.75\,(40)\right] = 31.50 \\ V_{0,0} &= 0.95\left[(1-0.50)\,(4.50)+0.50\,(31.50)\right] \\ &= 17.10 \text{ USD} \end{aligned} \tag{4.5.2}$$

`backwardInductionValue` reproduces the three node rows `[[17.1], [4.5, 31.5], [0, 20, 40]]` in its independent reference test.

Knowledge check 4.5.1: Multi-period lattice

1. At a node, next-period claim values are 10 in the down state and 30 in the up state. The risk-neutral up weight is 0.4 and the one-period discount factor is 0.95. Calculate the current node value.

2. At a bond-tree node, next-period ex-coupon values are 90 and 110, both states also pay a coupon of 5, the risk-neutral up weight is 0.25, and the node discount factor is 0.96. Calculate the current node value.

Answers: end of the chapter.

4.5.4 Sources

The replication, risk-neutral weighting, and backward-induction construction follow Hull, Chapter 12 §§12.1–12.3 [1]. The node discount factors are model inputs. Calibration and short-rate dynamics are outside this lesson.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012).

Answers to the knowledge checks

Knowledge check 4.5.1: Multi-period lattice

1. 17.1 (accepted within ±0.000001). The node value is 0.95 × (0.6 × 10 + 0.4 × 30) = 17.1.

2. 96 (accepted within ±0.000001). Add the coupon before averaging: 0.96 × [0.75 × 95 + 0.25 × 115] = 96.

Appendix A

Notation

The symbols of this book, in the order they first appear.

N **Forward contract quantity**. Positive number of underlying units delivered under the forward contract. Contract value and payoff scale linearly with this quantity. *Units:* units of underlying. *In Chapter 4.1 only.*

$$N > 0$$

$\Pi^{\text{long}}_{T_{\text{fwd}}}$ **Long forward expiry payoff**. Signed delivery-time value received by the long after exchanging the fixed delivery payment for the underlying. Per unit it equals underlying value at delivery minus the contractual delivery price. *Units:* stated currency at forward delivery time. *In Chapter 4.1 only.*

$A(0,t)$ **Accumulation factor**. Grows one current unit over a stated future horizon under the selected rate model. *Units:* dimensionless currency-units per current currency-unit.

m **Compounding frequency**. Number of equal compounding periods per year under the stated rate convention, a positive integer fixed by the model convention. *Units:* compounding periods per year.

S_t **Derivative underlying value**. Value at model time t of one unit of the asset or claim named as the derivative's underlying; a positive quoted value, while a position in the underlying carries its own signed quantity. *Units:* stated currency per unit of underlying at model time.

$D(0,t)$ **Discount factor**. Converts one deterministic future unit into value at valuation time. *Units:* current currency-units per future currency-unit.

V^{fwd}_t **Forward contract value**. Signed current value of an existing forward from the named counterparty's perspective; positive to the long when the current forward price exceeds the contract's fixed delivery price under the lesson model. *Units:* stated currency at model time.

K_{fwd} **Forward delivery price**. Contractual currency amount per unit of underlying paid by the long at delivery and received by the short. *Units:* delivery-time currency per unit of underlying.

T_{fwd} **Forward delivery time**. Future model time when the forward counterparties exchange the underlying and delivery payment; a contract date shared by the long and short, not the underlying asset's maturity. *Units:* model-years from the stated valuation time.

$F_{t,T_{\text{fwd}}}$ **Forward price**. Delivery price that would give a newly struck forward for the stated delivery time zero current value; a quoted contract rate rather than a cash amount received at quotation time. *Units:* delivery-time currency per unit of underlying.

$j^{(m)}$ **Nominal annual rate**. Annualized rate quote that must be paired with its compounding frequency. *Units:* decimal rate per year.

r_m **Periodic rate**. Rate applied once in each compounding period under the stated convention, derived from the stated nominal annual quote in this model. *Units:* decimal per compounding period.

I_0 **Underlying income present value**. Valuation-time value of deterministic cash income paid by the underlying after valuation and before the contract horizon, forward delivery or the matched option expiry. It is subtracted from spot value because the forward or option position does not receive that income. *Units:* stated currency at valuation time per unit of underlying.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

$\Pi^{\text{call}}_{T_{\text{opt}}}$ **Long call expiry payoff**. Non-negative amount received by the holder of a European call at expiry. Per unit it is the positive part of underlying expiry value minus strike. *Units:* stated currency at option expiry. *In Chapter 4.2 only.*

$\Pi^{\text{put}}_{T_{\text{opt}}}$ **Long put expiry payoff**. Non-negative amount received by the holder of a European put at expiry. Per unit it is the positive part of strike minus underlying expiry value. *Units:* stated currency at option expiry. *In Chapter 4.2 only.*

$G_{T_{\text{opt}}}$ **Option holder expiry profit**. Expiry payoff less the premium carried to expiry under the explicitly stated financing convention. In the simplest examples financing is ignored, so the initially paid premium is subtracted directly. *Units:* stated currency at option expiry. *In Chapter 4.2 only.*

π_0 **Option premium paid**. Positive valuation-time amount paid by the option holder to acquire the contractual right. Its financing convention must be stated before converting an expiry payoff into profit. *Units:* stated currency at premium payment time. *In Chapter 4.2 only.*

c_t **European call value**. Current non-negative value to the holder of a European call under the stated model, before any financing or transaction costs. *Units:* stated currency at model time.

T_{opt} **Option expiry time**. Future model time when a European option's exercise decision and payoff are determined; a contract date shared by holder and writer, distinct from a bond maturity. *Units:* model-years from the stated valuation time.

K **Option strike price**. Contractual price per unit of underlying used to determine the option's exercise payoff; a positive contractual amount, with payoff signs depending on call or put and holder or writer perspective. *Units:* expiry-time currency per unit of underlying.

p_t **European put value**. Current non-negative value to the holder of a European put under the stated model, before any financing or transaction costs. *Units:* stated currency at model time.

$\widetilde{F}_{0,T_{\text{opt}}}$ **Prepaid forward value**. Current value of receiving the underlying at expiry with no delivery payment then. Under the known-income model it equals spot value less the present

value of income paid before expiry. *Units:* stated currency at valuation time per unit of underlying. *In Chapter 4.3 only.*

V_0 **Binomial claim value**. Time-zero value obtained identically from the replicating portfolio and risk-neutral expected payoff. *Units:* stated currency at valuation time. *In Chapter 4.4 only.*

X^{down} **Binomial down claim payoff**. Signed expiry payoff of the claim when the underlying takes its down-state value. *Units:* stated currency at expiry. *In Chapter 4.4 only.*

S^{down} **Binomial down underlying value**. The lower of the two possible underlying values at the one-period expiry. *Units:* stated currency per unit of underlying at expiry. *In Chapter 4.4 only.*

Δ **Binomial hedge units**. Signed units of underlying in the portfolio that replicates the two claim payoffs. *Units:* units of underlying per claim. *In Chapter 4.4 only.*

$M_{T_{\text{opt}}}$ **Binomial risk-free expiry cash**. Signed expiry cash amount held with the underlying units to reproduce both claim payoffs. *Units:* stated currency at expiry. *In Chapter 4.4 only.*

q **Binomial risk-neutral up weight**. Model-implied pricing probability assigned to the up state by the no-arbitrage tree. It is dimensionless and lies between zero and one when the tree is arbitrage-consistent. *Units:* probability between zero and one. *In Chapter 4.4 only.*

X^{up} **Binomial up claim payoff**. Signed expiry payoff of the claim when the underlying takes its up-state value. *Units:* stated currency at expiry. *In Chapter 4.4 only.*

S^{up} **Binomial up underlying value**. One of the two possible underlying values at the one-period expiry. *Units:* stated currency per unit of underlying at expiry. *In Chapter 4.4 only.*

$\mathbb{E}$ **Expectation**. Operator returning the average of a random quantity, each outcome weighted by its probability under a stated measure; a superscript names that measure when more than one is in play. *Units:* value units.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

$\mathbb{Q}$ **Risk-neutral probability measure**. Gives model pricing weights under which discounted traded prices satisfy the martingale condition; it prices payoffs relative to a stated numeraire and is not a forecast of actual event frequencies. *Units:* dimensionless probability weights between zero and one.

$CF_{i+1,j}$ **Lattice next cash flow**. Signed cash paid at a successor node immediately before its continuation value is measured. *Units:* stated currency at the successor time. *In Chapter 4.5 only.*

$d_{i,j}$ **Lattice node discount factor**. One-period discount factor at a lattice node: currency at the node per one unit of currency at either successor node one step later. *Units:* current-node currency per next-time currency.

$q_{i,j}$ **Lattice node up weight**. Pricing probability assigned to the up successor, conditional on the current node; in a lattice with issuer default, also conditional on survival to the next lattice time. *Units:* probability between zero and one.

$V_{i,j}$ **Lattice node value**. Claim value at time row i and state node j obtained by one-period backward induction from its successor nodes, conditional on reaching that node under the input pricing lattice. *Units:* stated currency at the node time.

j **Lattice state index**. Integer label for one state node within a time row of a finite recombining lattice; a bookkeeping label under the stated successor ordering, not a probability or state value. *Units:* dimensionless integer index.

i **Lattice time index**. Integer label for one time row in a finite recombining valuation lattice; a bookkeeping label, not a model-year time or currency amount. *Units:* dimensionless integer index.

Appendix B

Abbreviations

CDS Credit default swap, the credit derivative the CDS lessons define and value.

CDX The name of a family of standard credit default swap indices, each a standard portfolio of single-name contracts.

DTS Duration times spread, a spread-risk measure for bonds.

FINRA Financial Industry Regulatory Authority.

IMM International Money Market. In the CDS lessons, IMM dates are the standard maturity dates on the twentieth of March, June, September, and December.

ISDA International Swaps and Derivatives Association.

ISO International Organization for Standardization.

JTD Jump to default, the loss on an immediate default of the reference entity.

UTC Coordinated Universal Time, the time standard the date arithmetic counts calendar days in.