Chapter 1.7

Risk-neutral pricing is not a risk-free probability

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/foundations/risk-neutral-pricing/

1.7.1 What you will be able to do

After this lesson, you should be able to:

- distinguish a real-world probability from a risk-neutral pricing weight;
- explain why changing measure does not remove uncertainty or volatility;
- calculate an expected discounted payoff in a finite-state model;
- state why a simple CDS model can omit a separate volatility input without claiming that volatility never matters.

1.7.2 Two measures for two different uses

Valuation is at valuation-time 0. Let the local risk-neutral-pricing-horizon T be a future time in exact model-years, and let the terminal-random-payoff X_T be the signed USD amount that a claim holder receives or pays at T.

A probability measure assigns mutually consistent weights to events. The real-world-probability-measure $\mathbb{P}$ describes the modeled probabilities of events in the real world. It is the measure for a forecast. The risk-neutral-probability-measure $\mathbb{Q}$ gives the weights of an arbitrage-free pricing model. So the same event can have different probabilities under $\mathbb{P}$ and $\mathbb{Q}$. Shreve makes this distinction between the actual and the risk-neutral measure in the introduction of the change of measure [1]. Hull makes the same distinction between forecasting and valuation [2].

> **There is no "risk-free measure" in this model.** The measure is **risk-neutral**. "Risk-free" describes the cash account and discounting convention used as the pricing benchmark. It does not describe the event probabilities, and it does not make the payoff safe.

The name "risk-neutral" also does not assume that every investor is indifferent to risk. Risk preferences and risk premia are among the reasons why $\mathbb{P}$ and $\mathbb{Q}$ differ. The pricing calculation uses $\mathbb{Q}$ after $\mathbb{Q}$ has been selected or calibrated.

1.7.3 From a partition to a price

In a no-arbitrage model with the cash account as numeraire, $\mathbb{Q}$ is chosen so that discounted traded prices are martingales. In words: after discounting with the cash account, today's traded price equals the conditional $\mathbb{Q}$-expectation of its discounted future value. Shreve states the martingale condition and its discounted-payoff consequence [3]. Tuckman and Serrat give the same expected-discounted-value construction in a finite-state fixed-income setting [4].

Write the expectation under the risk-neutral-probability-measure as $\mathbb{E}^{\mathbb{Q}}$. It is the same averaging operator, taken with the pricing weights of $\mathbb{Q}$. Use the local risk-neutral-scenario-index j and risk-neutral-scenario-count m. Let x_j be the local risk-neutral-scenario-payoff, in USD at T, and let q_j be its dimensionless risk-neutral-scenario-probability. For disjoint, exhaustive scenarios on which the payoff is constant, the risk-neutral expected payoff is:

$$\mathbb{E}^{\mathbb{Q}}[X_T] = \sum_{j=1}^{m} x_j q_j, \qquad q_j \geq 0, \qquad \sum_{j=1}^{m} q_j = 1 \tag{1.7.1}$$

This expected payoff is in USD at T. It is not a present value. Let the input discount-factor be $D(0,T)$, in USD at time zero per USD at T. The claim's present-value is the discounted risk-neutral expected payoff:

$$PV_0 = \mathbb{E}^{\mathbb{Q}}[D\,(0,T)\,X_T] = D(0,T)\mathbb{E}^{\mathbb{Q}}[X_T] = D(0,T)\sum_{j=1}^{m} x_j q_j \tag{1.7.2}$$

The second equality uses this lesson's assumption that $D(0,T)$ is deterministic. If the discount factor is random, it must in general stay inside the expectation. The expectation of a product equals the product of the expectations only under additional assumptions.

1.7.4 Worked example

Suppose $D(0,T) = 0.95$ and the claim has two disjoint, exhaustive outcomes:

Table 1.7.1 **Scenarios of the worked example.** Each scenario's risk-neutral probability, payoff at the horizon, and contribution to the risk-neutral expected payoff, in USD at the horizon.

Scenario j	Risk-neutral probability q_j	Payoff x_j, USD at T	Contribution $q_j x_j$, USD at T
1	0.25	120	30.00
2	0.75	40	30.00

The risk-neutral expected payoff (1.7.1) and time-zero value (1.7.2) are

$$\mathbb{E}^{\mathbb{Q}}[X_T] = 0.25(120) + 0.75(40) = \text{USD } 60 \text{ at } T \tag{1.7.3}$$

$$PV_0 = 0.95(60) = \text{USD } 57 \text{ at time } 0 \tag{1.7.4}$$

No $\mathbb{P}$ probabilities enter this price, because the example states the $\mathbb{Q}$ pricing weights as inputs. An estimate of the real-world expected payoff would be a different calculation.

1.7.5 Uncertainty remains

A change from $\mathbb{P}$ to $\mathbb{Q}$ changes the weights of the modeled outcomes. It does not remove the randomness. In the standard change-of-measure result for a diffusion, the drift changes and the diffusion coefficient stays the same [5]. So volatility remains a model input for nonlinear claims such as options. A restricted linear model can omit a separate volatility input for reasons specific to its product. The lesson on that product states those reasons.

1.7.6 Advanced boundary: uniqueness

The pricing measure need not be unique. Under the usual technical conditions, the absence of arbitrage is tied to the existence of a suitable martingale measure. Uniqueness of the measure requires a stronger condition: the modeled market must be complete. This section is a boundary note, and the assessment of this lesson does not test it. Shreve develops the existence and uniqueness distinction [6].

1.7.7 Check your understanding

These checks separate interpretation from arithmetic and include unfamiliar scenario weights and a signed negative payoff.

Knowledge check 1.7.1: Risk-neutral pricing

1. Which statement correctly distinguishes a real-world probability measure P from a risk-neutral probability measure Q in this lesson?
   (a) P describes modeled real-world frequencies; Q gives no-arbitrage pricing weights
   (b) P and Q must assign the same probability to every event
   (c) Q assigns certainty to every traded payoff
2. An analyst says, 'After changing from P to Q, uncertainty disappears and volatility is absorbed by the measure.' Which response is correct for the standard diffusion result taught here?
   (a) The change reweights outcomes and changes drift, but it does not remove randomness or the diffusion volatility
   (b) The change makes every future payoff deterministic
   (c) The change sets every volatility parameter to zero
3. At T, a signed payoff is USD 120 with Q-probability 0.70 and USD 20 with Q-probability 0.30. The deterministic discount factor is $D(0,T) = 0.96$. What is its time-zero risk-neutral value in USD?
4. A contract has signed terminal payoffs of USD 200, USD 50, and USD -40 in three disjoint exhaustive scenarios. Their Q-probabilities are 0.20, 0.50, and 0.30, and $D(0,T) = 0.94$. What is the time-zero risk-neutral value in USD?

Answers: *end of the chapter.*

1.7.8 Model boundary and review note

This lesson uses one terminal payoff, a finite scenario partition, deterministic risk-free discounting, and input $\mathbb{Q}$ weights. It does not estimate $\mathbb{P}$, calibrate $\mathbb{Q}$, generate a yield curve, treat a stochastic numeraire, establish market completeness, or value counterparty, liquidity, funding, or option effects.

All source records, notation, formulas, examples, and answer keys remain `draft` pending independent human source, quantitative, and editorial review.

References

[1] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 1 §1.6, printed pp. 32-35.

[2] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 12 §12.2, printed pp. 257-260; Ch. 14 §14.7, printed pp. 311-313.

[3] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 5 §5.2.2, printed pp. 214-217, eqs. 5.2.22-5.2.24; §5.2.4, printed pp. 218-219, eqs. 5.2.29-5.2.31.

[4] Tuckman & Serrat, Fixed Income Securities: Tools for Today's Markets (4th ed., 2022). Ch. 7 §7.3, printed pp. 182-184, eqs. 7.7-7.8.

[5] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 5 §5.2.2, printed pp. 214-217, especially eqs. 5.2.22-5.2.24.

[6] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 5 §§5.4.3-5.4.4, printed pp. 228-233.

Notation

T **Risk-neutral pricing horizon**. Future model time at which the one terminal payoff is delivered. It is measured in exact model-years from valuation time zero and is not a calendar date. *Units:* years from valuation time.

$$T > 0$$

m **Risk-neutral scenario count**. Counts the disjoint and exhaustive scenarios in the finite pricing partition. It is a positive integer and does not measure time or money.

$$m \in \mathbb{N}$$

j **Risk-neutral scenario index**. Selects one scenario in the finite pricing partition. It is a bookkeeping label rather than a probability or time.

$$j \in \{1, \dots, m\}$$

x_j **Risk-neutral scenario payoff**. Realized signed terminal amount in one scenario. Positive means received and negative means paid by the claim holder at the horizon. *Units:* stated currency at the horizon.

q_j **Risk-neutral scenario probability**. Pricing weight assigned by the risk-neutral measure to one scenario. The weights are dimensionless, non-negative, and sum to one over the finite partition. *Units:* dimensionless probability.

$$q_j \geq 0, \quad \sum_{j=1}^{m} q_j = 1$$

$A(0,t)$ **Accumulation factor**. Grows one current unit over a stated future horizon under the selected rate model. *Units:* dimensionless currency-units per current currency-unit.

m **Compounding frequency**. Number of equal compounding periods per year under the stated rate convention, a positive integer fixed by the model convention. *Units:* compounding periods per year.

$D(0,t)$ **Discount factor**. Converts one deterministic future unit into value at valuation time. *Units:* current currency-units per future currency-unit.

$\mathbb{E}$ **Expectation**. Operator returning the average of a random quantity, each outcome weighted by its probability under a stated measure; a superscript names that measure when more than one is in play. *Units:* value units.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

$j^{(m)}$ **Nominal annual rate**. Annualized rate quote that must be paired with its compounding frequency. *Units:* decimal rate per year.

t_k **Payment time**. Time from the valuation date to one scheduled cash-flow event; a time coordinate, not a calendar date or payment index. *Units:* years from the valuation date.

r_m **Periodic rate**. Rate applied once in each compounding period under the stated convention, derived from the stated nominal annual quote in this model. *Units:* decimal per compounding period.

PV_0 **Present value**. Combines dated signed cash flows into one value at valuation time, using the same holder perspective as the signed cash flows. *Units:* stated currency at the valuation time.

$\mathbb{P}$ **Real-world probability measure**. Assigns modeled probabilities intended to describe actual-world event likelihoods; used for forecasting and statistical statements under the stated real-world model. *Units:* dimensionless probability weights between zero and one.

$\mathbb{Q}$ **Risk-neutral probability measure**. Gives model pricing weights under which discounted traded prices satisfy the martingale condition; it prices payoffs relative to a stated numeraire and is not a forecast of actual event frequencies. *Units:* dimensionless probability weights between zero and one.

CF_k **Signed cash flow**. Amount received or paid at one event from the stated holder perspective; positive means received and negative means paid by that holder. *Units:* stated currency units.

X_T **Terminal random payoff**. Signed amount delivered by a claim at the stated future horizon, before its outcome is known; positive means received and negative means paid by the claim holder. *Units:* stated currency at the future horizon.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

Answers to the knowledge checks

Knowledge check 1.7.1: Risk-neutral pricing

1. **(a)** P describes modeled real-world frequencies; Q gives no-arbitrage pricing weights. P and Q can weight the same outcomes differently. P is the modeled real-world measure; Q is selected so discounted traded prices satisfy the model's no-arbitrage pricing condition.

2. **(a)** The change reweights outcomes and changes drift, but it does not remove randomness or the diffusion volatility. Risk-neutral valuation changes the probability weights used for pricing. In the standard change-of-measure result, the drift changes while the diffusion coefficient remains; uncertainty is still present.

3. 86.4 (accepted within ± 0.01). First compute the Q-expectation: $0.70 \times 120 + 0.30 \times 20 = 90$ future USD. Then discount: $0.96 \times 90 =$ USD 86.40 at time zero.

4. 49.82 (accepted within ± 0.01). The Q-expected terminal payoff is $0.20 \times 200 + 0.50 \times 50 + 0.30 \times (-40) =$ USD 53. Discounting gives $0.94 \times 53 =$ USD 49.82.