Chapter 1.2

Events, conditional probability, and expectation

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/foundations/probability-events-and-expectation/

1.2.1 What you will be able to do

After this lesson, you should be able to:

- identify events that form a disjoint and exhaustive partition;
- calculate a conditional probability;
- calculate an expectation when a random value is constant on each event;
- use conditional event means when the value varies inside an event.

1.2.2 Outcomes, events, and probabilities

An **outcome** is one possible result of a modeled experiment. An **event** is a specified set of outcomes. Use the local probability-event A for one such set and the local probability-measure $\mathbb{P}$ for the rule that assigns probabilities to events. In this lesson, $\mathbb{P}$ is a general probability measure. A later finance lesson uses $\mathbb{P}$ only for real-world probabilities and introduces a different symbol for pricing probabilities. This lesson does not choose between the two.

In this lesson, a finite partition is a finite list of events that are:

- **disjoint:** no outcome belongs to two different partition events; and
- **exhaustive:** every possible outcome belongs to one of them.

Use the local probability-event-index j to select an event and the local probability-partition-size m to count the events. For a disjoint and exhaustive list $A_1, \dots, A_m$, additivity of the probability measure gives the sum of the event probabilities:

$$\sum_{j=1}^{m} \mathbb{P}(A_j) = 1 \tag{1.2.1}$$

If the events overlap, the sum of their probabilities can count some outcomes twice. If the events are not exhaustive, the sum omits some outcomes. These event and additivity definitions follow Shreve's probability-space treatment[1].

1.2.3 Conditional probability restricts the outcomes

Suppose the question asks for the probability of event A **given** another event. Write the conditioning event as B. When $\mathbb{P}(B) > 0$, the conditional probability of A given B is defined as:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \tag{1.2.2}$$

The denominator is the probability of the outcomes that the condition "given B" keeps. For example, suppose 30% of all outcomes are in B, and 12% are in both A and B. Then the conditional probability is:

$$\mathbb{P}(A \mid B) = \frac{0.12}{0.30} = 0.40 \tag{1.2.3}$$

In a default model, consider the probability of "default during this period given survival to its start." Its denominator is the probability of survival to the start of the period. This conditional probability is different from the unconditional probability of default by a horizon. Shreve develops conditioning as averaging within the information or event retained by the condition[2].

1.2.4 Expectation from disjoint events

Use the local probability-random-value X for a numerical value determined by the outcome and the expectation operator $\mathbb{E}$ for its probability-weighted average. In this lesson every random value is constant on each event of a finite partition, so the general integral definition of $\mathbb{E}$ reduces to the weighted sum below. The expectation has the same unit as X; a random USD payment has an expected value in USD, not in percent.

This section gives two formulas. The first formula requires X to be constant on each event. The second formula does not.

1.2.4.1 Exact value on every event

First suppose X is constant on every partition event. Use the local probability-event-value x_j for its value on A_j. Then the expectation is the sum of the event values, each multiplied by the probability of its event:

$$\mathbb{E}[X] = \sum_{j=1}^{m} x_j \mathbb{P}(A_j), \qquad X = x_j \text{ on } A_j \tag{1.2.4}$$

Shreve states the expectation of a finite-valued random variable in the form of (1.2.4)[3].

1.2.4.2 Value varies inside an event

If X is not constant inside A_j, one representative value is in general not exact. For each event with positive probability, replace the value with the conditional mean of X inside that event. The expectation is then:

$$\mathbb{E}[X] = \sum_{j=1}^{m} \mathbb{P}(A_j)\mathbb{E}[X \mid A_j] \tag{1.2.5}$$

An event with zero probability contributes zero to the sum and can be omitted. Its conditional mean need not be defined.

The partition identity (1.2.5) follows by splitting an expectation across disjoint events and using the partial-averaging property of conditional expectation. Shreve gives the disjoint-event integral split and the unconditional-average property of conditional expectation[4].

Example 1.2.1: Expectation over disjoint events

Constant value on each event

Suppose X is a payment in USD. It equals USD 0, USD 50, or USD 100 on three disjoint exhaustive events with probabilities 0.10, 0.20, and 0.70.

Event	Constant payment, USD	Probability	Contribution, USD
A_1	0	0.10	0.00
A_2	50	0.20	10.00
A_3	100	0.70	70.00

The expectation is the sum of the contributions:

$$\mathbb{E}[X] = 0 \times 0.10 + 50 \times 0.20 + 100 \times 0.70 = \text{USD } 80$$

The expectation of USD 80 is a probability-weighted average. An expectation need not be one of the possible payments, and in this example it is not.

Conditional means inside events

Suppose three disjoint exhaustive events have probabilities 0.01, 0.02, and 0.97, and the payment varies inside each event. The conditional means of the payment are USD 40, USD 30, and USD 0. The exact expectation from these conditional means is:

$$\mathbb{E}[X] = 0.01 \times 40 + 0.02 \times 30 + 0.97 \times 0 = \text{USD } 1.00$$

USD 40 is the average payment inside the first event. The example does not claim that every outcome in the first event pays USD 40.

Default inside a period

For a CDS period, one event can be "default occurs in this period." A protection payment discounted at the actual default time varies within that event. Accrued premium also varies within the event, because a default near the end of the period accrues more premium than a default near its start.

So one midpoint value for the whole period is an approximation. The exact contribution of the event is the interval default probability multiplied by the conditional mean of the discounted payment inside the period. If the period is divided into smaller disjoint default-time intervals, the sum has more terms. In a continuous-time model, the limit of this sum is an integral over the default time. The later CDS lesson gives the hazard-rate probabilities and the cash-flow formulas for that integral.

1.2.5 Check your understanding

These items separately test events, conditioning, and both expectation forms. Each question stays collapsed until you open it; answers and explanations appear once you check.

Knowledge check 1.2.1: Events, conditioning, and expectation

1. Which description is an event in a probability model?
   (a) A specified set of possible outcomes
   (b) Only the number assigned as a probability
   (c) Only the average value of a random quantity
2. A one-year credit model allows at most one default. Which list is a disjoint and exhaustive partition of outcomes?
   (a) Default by 6 months; default after 6 months through 1 year; survive through 1 year
   (b) Default by 1 year; default after 6 months through 1 year; survive through 1 year
   (c) Default by 6 months; survive through 1 year
3. Events A and B satisfy P(A and B) = 0.12 and P(B) = 0.30. What is P(A given B)? Enter a decimal probability.
4. In a frequency example, 80 otherwise comparable names are alive at the start of a period and 2 of those names default during it. What conditional default fraction does the sample imply for that period? Enter a decimal.
5. A random payoff is constant at 0, 50, and 100 units on three disjoint exhaustive events with probabilities 0.10, 0.20, and 0.70. What is its expected payoff in units?
6. Three disjoint exhaustive events have probabilities 0.01, 0.02, and 0.97. A payment varies inside the events, but its conditional mean in each event is 40, 30, and 0 USD respectively. What is the unconditional expected payment in USD?

Answers: end of the chapter.

1.2.6 Model boundary and review note

This lesson teaches finite partitions and the idea of the limit of a refinement. It does not construct probability spaces, prove measure-theoretic integration, estimate probabilities from data, calibrate risk-neutral probabilities, or evaluate the default-time integrals used by a CDS model. A later lesson distinguishes real-world and risk-neutral measures.

The lesson paraphrases the cited textbook; it remains `draft` pending human verification of the printed locators, formulas, examples, and notation bindings.

References

[1] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 1 §1.1, printed pp. 1-3, Def. 1.1.2 and eqs. 1.1.2-1.1.5.

[2] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 2 §2.3, printed pp. 66-69, eqs. 2.3.4-2.3.17 and Def. 2.3.1.

[3] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 1 §1.3, printed pp. 13-18, Thms. 1.3.1 and 1.3.4.

[4] Shreve, Stochastic Calculus for Finance II: Continuous-Time Models (2004). Ch. 1 §1.3, printed p. 17, Remark 1.3.2; Ch. 2 §2.3, printed pp. 68-72, eqs. 2.3.16-2.3.17 and 2.3.25.

Notation

A **Probability event**. A specified set of possible outcomes in this lesson's probability model. An indexed form selects one event in a finite disjoint and exhaustive partition.

$$A_j$$

j **Probability event index**. Selects one event in the lesson's finite partition. The index is bookkeeping and has no probability or monetary unit.

$$j \in \{1, \dots, m\}$$

x_j **Probability event value**. The constant value taken by the random value on one partition event. This quantity can replace a conditional mean only when every outcome in the event gives the same value. *Units:* value units.

$\mathbb{P}$ **Probability measure**. Assigns each modeled event a number between zero and one. The symbol is page-local and generic; it does not yet select a real-world or risk-neutral interpretation. *Units:* probability between zero and one.

A_j **Probability partition event**. One event of the lesson's finite disjoint and exhaustive partition, selected by the event index.

m **Probability partition size**. Counts the events in the lesson's finite partition. The count is a positive integer and does not represent time.

$$m \in \mathbb{N}$$

X **Probability random value**. A numerical value determined by which modeled outcome occurs. Its unit is stated by the example, such as USD for a random payment. *Units:* value units.

$\mathbb{E}$ **Expectation**. Operator returning the average of a random quantity, each outcome weighted by its probability under a stated measure; a superscript names that measure when more than one is in play. *Units:* value units.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

Answers to the knowledge checks

Knowledge check 1.2.1: Events, conditioning, and expectation

1. **(a)** A specified set of possible outcomes. An event is a set of outcomes. A probability measure assigns a number to that set.

2. **(a)** Default by 6 months; default after 6 months through 1 year; survive through 1 year. The three correct events cannot occur together and cover every outcome in the stated one-default model. The second list overlaps, and the third omits defaults in the second half-year.

3. 0.4 (accepted within $\pm 1e-12$). Divide the intersection probability by the conditioning-event probability: 0.12 / 0.30 = 0.40.

4. 0.025 (accepted within $\pm 1e-12$). Condition on the 80 names alive at period start: 2 / 80 = 0.025, or 2.5%. This sample fraction is not by itself a calibrated pricing probability.

5. 80 (accepted within $\pm 1e-9$). Weight each event value by its probability: $0 \times 0.10 + 50 \times 0.20 + 100 \times 0.70 = 80$.

6. 1 (accepted within $\pm 1e-9$). Use event probability times conditional event mean: $0.01 \times 40 + 0.02 \times 30 + 0.97 \times 0 =$ USD 1.00.