Chapter 1.6

No-arbitrage and replication

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/foundations/no-arbitrage-and-replication/

1.6.1 What you will be able to do

After this lesson, you should be able to:

- explain why identical dated state-contingent cash flows have the same price under the stated assumptions;
- distinguish exact replication from matching only an expected cash flow;
- calculate a claim value from signed positions in a replicating portfolio.

1.6.2 Compare cash flows, not labels

The law of one price applies when two portfolios produce the same signed-cash-flow at every relevant future date and in every modeled state. The names of the portfolios can differ. What matters is the full set of dated cash flows, from the same perspective.

Suppose the two portfolios had different prices at valuation-time 0. In the frictionless model of this lesson, a trader could then buy the cheaper portfolio and sell the more expensive one. The future cash flows of the two positions cancel, so the trader keeps the initial difference in price. This arbitrage argument is the reason for the pricing rule. Tuckman and Serrat state the law of one price and connect deviations from it to an arbitrage trade built from a replicating portfolio[1].

Matching only the expected cash flow is not enough. A payment of USD 100 only in an up state is not the same cash flow as USD 100 only in a down state, even if the two states have equal probability.

1.6.3 Value the replicating portfolio

Use the local replication-security-index i to select one security and the local replication-security-count n to count them. Let the signed replication-security-quantity be a_i and the current replication-security-value be $V_{0,i}$. When this portfolio exactly matches the claim's future cash flows, the replicated-claim-value is the value of the portfolio:

$$V_0^{\text{claim}} = \sum_{i=1}^{n} a_i V_{0,i} \tag{1.6.1}$$

A positive a_i is a long position, and a negative a_i is a short position. The formula uses current values because the portfolio already matches the claim's cash flows at every future date and in every state.

Suppose a claim is replicated by two units of security A worth USD 12 each and one unit of security B worth USD 7. The value of the replicating portfolio, and therefore of the claim, is:

$$V_0^{\text{claim}} = 2(12) + 1(7) = \text{USD } 31 \tag{1.6.2}$$

1.6.4 Where replication is used next

Forward prices use cash-and-carry replication. Put-call parity compares two portfolios with identical expiry cash flows. A binomial option model constructs a state-by-state replicating portfolio. Risk-neutral valuation is another way to calculate that same no-arbitrage value inside a specified model.

Knowledge check 1.6.1: No-arbitrage and replication

1. Portfolio A and claim B have identical signed cash flows at every future date in every modeled state. Under the lesson's frictionless no-arbitrage assumptions, what must be true?
   (a) They must have the same current value
   (b) They need the same contract name but may have unrelated values
   (c) Only their average future cash flow needs the same value
2. Two portfolios have the same expected maturity cash flow, but one pays only in an up state and the other only in a down state. Does the law of one price force their current values to be equal?
   (a) No, matching an expectation does not replicate state-by-state cash flows
   (b) Yes, the same expected cash flow always implies the same value
   (c) Yes, sharing a maturity date is sufficient
3. A claim is exactly replicated by two units of security A worth USD 12 each and one unit of security B worth USD 7. What is the claim's no-arbitrage value in USD?
4. A claim is exactly replicated by buying three units of security A worth USD 18 each and shorting two units of security B worth USD 11 each. What is the claim's no-arbitrage value in USD?

Answers: *end of the chapter.*

1.6.5 Model boundary and review note

Real trades can have bid-ask spreads, different borrowing and lending rates, short-sale constraints, collateral, taxes, liquidity differences, and execution risk. These effects can make a textbook

arbitrage impossible to execute. This lesson sets up a frictionless pricing benchmark. It does not claim that every observed price difference is a riskless profit.

The source locator, notation, examples, and answer keys remain `draft` pending independent human review.

References

[1] Tuckman & Serrat, Fixed Income Securities: Tools for Today's Markets (4th ed., 2022). Ch. 1 §§1.3-1.4, printed pp. 53-57, law of one price and arbitrage replication.

Notation

V_0^{claim} **Replicated claim value**. Current value assigned to the claim whose future cash flows are exactly reproduced. Under the stated no-arbitrage assumptions it equals the signed current value of the replicating portfolio. *Units:* stated currency at valuation time.

n **Replication security count**. Counts the traded securities in the finite replicating portfolio. It is a positive integer rather than a time or cash amount.

$$n \in \mathbb{N}$$

i **Replication security index**. Selects one traded security in the finite replicating portfolio. It is a bookkeeping label and has no monetary unit.

$$i \in \{1, \dots, n\}$$

a_i **Replication security quantity**. Number of units of security i held in the replicating portfolio. A positive quantity is long and a negative quantity is short from the portfolio holder's perspective. *Units:* security units.

$V_{0,i}$ **Replication security value**. Current currency value per unit of one security used in the replicating portfolio. Every value uses the same valuation time and currency. *Units:* stated currency per security at valuation time.

t_k **Payment time**. Time from the valuation date to one scheduled cash-flow event; a time coordinate, not a calendar date or payment index. *Units:* years from the valuation date.

CF_k **Signed cash flow**. Amount received or paid at one event from the stated holder perspective; positive means received and negative means paid by that holder. *Units:* stated currency units.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

Answers to the knowledge checks

Knowledge check 1.6.1: No-arbitrage and replication

1. **(a)** They must have the same current value. If identical future cash flows had different current values, buying the cheaper package and selling the dearer one would create an initial receipt with offsetting future obligations under the stated assumptions.

2. **(a)** No, matching an expectation does not replicate state-by-state cash flows. Replication requires the same dated cash flow in every modeled state. Equal averages alone leave different state exposure.

3. 31 (accepted within ± 0.000001). The replicating portfolio costs $2 \times 12 + 1 \times 7 =$ USD 31, so the law of one price gives the claim the same value.

4. 32 (accepted within ± 0.000001). The signed portfolio value is 3×18 - $2 \times 11 =$ USD 32. The short position enters with a negative quantity.