Chapter 4.4

One-period binomial option valuation

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/one-period-binomial-option-valuation/

4.4.1 What you will be able to do

After this lesson, you should be able to:

- construct the two-position portfolio that matches both state payoffs;
- calculate the tree's risk-neutral up-state weight;
- show that replication and discounted risk-neutral expectation give the same value.

4.4.2 Two states, two traded positions

Let the underlying be worth S_0 at valuation-time and take exactly the local values binomial-up-underlying-value S^{up} or binomial-down-underlying-value S^{down} at the option-expiry-time T_{opt}. A claim pays local binomial-up-claim-payoff X^{up} or binomial-down-claim-payoff X^{down}.

The local binomial-hedge-units Δ is the number of units of the underlying that matches the difference between the two claim payoffs:

$$\Delta = \frac{X^{\text{up}} - X^{\text{down}}}{S^{\text{up}} - S^{\text{down}}} \tag{4.4.1}$$

The local binomial-risk-free-expiry-cash is then chosen so that the portfolio matches the claim payoff in the down state:

$$M_{T_{\text{opt}}} = X^{\text{down}} - \Delta\, S^{\text{down}} \tag{4.4.2}$$

So the local binomial-claim-value is the value of the replicating portfolio at time zero:

$$V_0 = \Delta\, S_0 + D(0, T_{\text{opt}}) M_{T_{\text{opt}}} \tag{4.4.3}$$

4.4.3 The same value with risk-neutral weights

The local binomial-risk-neutral-up-weight q is chosen so that the model reproduces the current value of the underlying:

$$q = \frac{S_0/D(0,T_{\text{opt}}) - S^{\text{down}}}{S^{\text{up}} - S^{\text{down}}} \tag{4.4.4}$$

Under the risk-neutral-probability-measure $\mathbb{Q}$, the claim value is the discounted risk-neutral expected payoff:

$$V_0 = D(0,T_{\text{opt}}) \left(q\, X^{\text{up}} + (1-q)\, X^{\text{down}} \right) \tag{4.4.5}$$

Hull derives the one-step hedge and then shows that its no-arbitrage value equals the discounted risk-neutral expected payoff [1].

4.4.4 Worked call example

Let $S_0 = 100$, $S^{\text{up}} = 120$, $S^{\text{down}} = 80$, $D(0,T_{\text{opt}}) = 1/1.05$, and let the option-strike-price be $K = 100$. The call payoffs are $X^{\text{up}} = 20$ and $X^{\text{down}} = 0$.

Then $\Delta = 0.5$, $M_{T_{\text{opt}}} = -40$, and $q = 0.625$. Both methods give the same claim value:

$$V_0 = 0.5(100) + \frac{-40}{1.05} = \frac{0.625(20) + 0.375(0)}{1.05} = 11.9047619 \tag{4.4.6}$$

The negative cash position is borrowing. It is not a negative option value.

4.4.5 From one period to many periods

Repeating this valuation one period at a time, from expiry backward, is backward induction on a multi-period tree. The later bond-option lessons need a discount factor at each node and an additional state for survival and default. An option with a knockout triggered by default cannot be valued from the distribution at expiry alone, because the option can terminate before expiry.

Knowledge check 4.4.1: One-period binomial option valuation

1. A no-income underlying is worth USD 100 now and will be USD 120 or USD 80 at expiry. D(0,T) = 1/1.05. A call with strike USD 100 pays USD 20 or USD 0. The model-implied Q up-weight is 0.625. What is the call value in USD?

2. In the same tree, a put with strike USD 100 pays USD 0 in the up state and USD 20 in the down state. The Q up-weight is 0.625 and D(0,T) = 1/1.05. What is the put value in USD?

3. A claim pays USD 20 when the underlying is USD 120 and USD 0 when it is USD 80. How many underlying units are required in its one-period replicating portfolio?

4. A claim pays USD 10 when the underlying is USD 130 and USD -2 when it is USD 90. Its hedge quantity is 0.3 underlying units. What risk-free cash amount is required at expiry to reproduce the down-state payoff, in USD?

Answers: *end of the chapter.*

4.4.6 Model boundary and review note

This complete two-state model has one non-income-paying traded underlying, one risk-free account, and deterministic discounting. It does not estimate real-world probabilities or volatility. It does not model multiple time steps, stochastic rates, default, or transaction and funding effects.

The two valuation methods are implemented together in a pure tested domain module. The hand-calculated answer and implementation both remain subject to independent human quantitative review.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 12 §§12.1-12.2, printed pp. 253-259, one-step binomial replication and risk-neutral valuation.

Notation

V_0 **Binomial claim value**. Time-zero value obtained identically from the replicating portfolio and risk-neutral expected payoff. *Units:* stated currency at valuation time.

X^{down} **Binomial down claim payoff**. Signed expiry payoff of the claim when the underlying takes its down-state value. *Units:* stated currency at expiry.

S^{down} **Binomial down underlying value**. The lower of the two possible underlying values at the one-period expiry. *Units:* stated currency per unit of underlying at expiry.

Δ **Binomial hedge units**. Signed units of underlying in the portfolio that replicates the two claim payoffs. *Units:* units of underlying per claim.

$M_{T_{\text{opt}}}$ **Binomial risk-free expiry cash**. Signed expiry cash amount held with the underlying units to reproduce both claim payoffs. *Units:* stated currency at expiry.

q **Binomial risk-neutral up weight**. Model-implied pricing probability assigned to the up state by the no-arbitrage tree. It is dimensionless and lies between zero and one when the tree is arbitrage-consistent. *Units:* probability between zero and one.

X^{up} **Binomial up claim payoff**. Signed expiry payoff of the claim when the underlying takes its up-state value. *Units:* stated currency at expiry.

S^{up} **Binomial up underlying value**. One of the two possible underlying values at the one-period expiry. *Units:* stated currency per unit of underlying at expiry.

$A(0,t)$ **Accumulation factor**. Grows one current unit over a stated future horizon under the selected rate model. *Units:* dimensionless currency-units per current currency-unit.

m **Compounding frequency**. Number of equal compounding periods per year under the stated rate convention, a positive integer fixed by the model convention. *Units:* compounding periods per year.

S_t **Derivative underlying value**. Value at model time t of one unit of the asset or claim named as the derivative's underlying; a positive quoted value, while a position in the underlying carries its own signed quantity. *Units:* stated currency per unit of underlying at model time.

$D(0,t)$ **Discount factor**. Converts one deterministic future unit into value at valuation time. *Units:* current currency-units per future currency-unit.

$\mathbb{E}$ **Expectation**. Operator returning the average of a random quantity, each outcome weighted by its probability under a stated measure; a superscript names that measure when more than one is in play. *Units:* value units.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

$j^{(m)}$ **Nominal annual rate**. Annualized rate quote that must be paired with its compounding frequency. *Units:* decimal rate per year.

T_{opt} **Option expiry time**. Future model time when a European option's exercise decision and payoff are determined; a contract date shared by holder and writer, distinct from a bond maturity. *Units:* model-years from the stated valuation time.

K **Option strike price**. Contractual price per unit of underlying used to determine the option's exercise payoff; a positive contractual amount, with payoff signs depending on call or put and holder or writer perspective. *Units:* expiry-time currency per unit of underlying.

r_m **Periodic rate**. Rate applied once in each compounding period under the stated convention, derived from the stated nominal annual quote in this model. *Units:* decimal per compounding period.

$\mathbb{Q}$ **Risk-neutral probability measure**. Gives model pricing weights under which discounted traded prices satisfy the martingale condition; it prices payoffs relative to a stated numeraire and is not a forecast of actual event frequencies. *Units:* dimensionless probability weights between zero and one.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

Answers to the knowledge checks

Knowledge check 4.4.1: One-period binomial option valuation

1. 11.9047619048 (accepted within ± 0.000001). Discount the Q-weighted payoff: $(1/1.05) \times [0.625 \times 20 + 0.375 \times 0]$ = USD 11.9047619048.

2. 7.1428571429 (accepted within ± 0.000001). The Q down-weight is 0.375. Discounting gives $(1/1.05) \times [0.625 \times 0 + 0.375 \times 20]$ = USD 7.1428571429.

3. 0.5 (accepted within ± 0.000001). The hedge quantity is the payoff difference divided by the underlying-value difference: (20 - 0) / (120 - 80) = 0.5.

4. -29 (accepted within ± 0.000001). Cash at expiry is $M_{T_{\text{opt}}} = X^{\text{down}} - \Delta\, S^{\text{down}} = -2 - 0.3 \times 90 =$ USD -29. A negative amount is borrowing.