Chapter 4.5

Multi-period lattice valuation

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/derivatives/multiperiod-lattice-valuation/

4.5.1 Repeat the one-period argument backward

The one-period binomial lesson valued two terminal payoffs. A multi-period lattice applies the same pricing step at every node, beginning with the last row and moving backward to valuation-time.

At the node with lattice-time-index i and lattice-state-index j, let the lattice-node-up-weight be $q_{i,j}$ and the lattice-node-discount-factor be $d_{i,j}$. Let the local next-date cash flow $CF_{i+1,j}$ be paid at the successor node $(i+1, j)$; the node value $V_{i+1,j}$ excludes it. The lattice-node-value at node (i, j) is the discounted risk-neutral expectation of the successor cash flows plus the successor node values:

$$V_{i,j} = d_{i,j}\left[\left(1 - q_{i,j}\right)\left(CF_{i+1,j} + V_{i+1,j}\right) + q_{i,j}\left(CF_{i+1,j+1} + V_{i+1,j+1}\right)\right] \tag{4.5.1}$$

The two successors of node (i, j) are the nodes $(i+1, j)$ and $(i+1, j+1)$. The labels "up" and "down" are conventions. The input up weights and values must use the same order.

4.5.2 Timing is part of the model

In (4.5.1), each next-date cash flow is added to the value of its successor node before discounting. So the value of a coupon bond at a node is ex-coupon: it excludes the coupon paid at that node. A change of the order of exercise and coupon payment changes the contract, and it can change the option payoff.

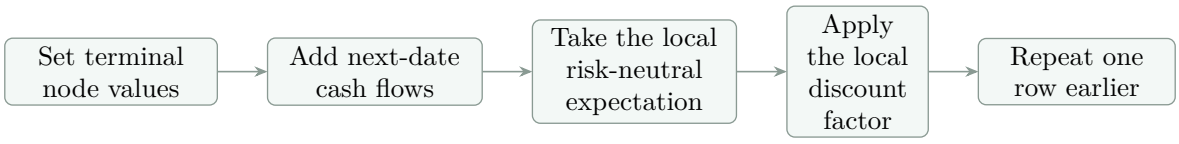


Diagram 4.5.1 **Backward-induction order.** Start from terminal node values. At each earlier row, add any next-date cash flow to each successor value, form the local risk-neutral weighted average, multiply by the local discount factor, and repeat.

4.5.3 Worked example

The terminal values are USD 0, USD 20, and USD 40. At time index 1, both nodes have discount factor 0.90, and their up weights are 0.25 and 0.75. At time index 0, the discount factor is 0.95 and the up weight is 0.50. Applying (4.5.1) row by row gives the node values:

$$\begin{aligned} V_{1,0} &= 0.90\left[(1-0.25)\,(0)+0.25\,(20)\right] = 4.50 \\ V_{1,1} &= 0.90\left[(1-0.75)\,(20)+0.75\,(40)\right] = 31.50 \\ V_{0,0} &= 0.95\left[(1-0.50)\,(4.50)+0.50\,(31.50)\right] \\ &= 17.10 \text{ USD} \end{aligned} \tag{4.5.2}$$

`backwardInductionValue` reproduces the three node rows `[[17.1], [4.5, 31.5], [0, 20, 40]]` in its independent reference test.

Knowledge check 4.5.1: Multi-period lattice

1. At a node, next-period claim values are 10 in the down state and 30 in the up state. The risk-neutral up weight is 0.4 and the one-period discount factor is 0.95. Calculate the current node value.

2. At a bond-tree node, next-period ex-coupon values are 90 and 110, both states also pay a coupon of 5, the risk-neutral up weight is 0.25, and the node discount factor is 0.96. Calculate the current node value.

Answers: end of the chapter.

4.5.4 Sources

The replication, risk-neutral weighting, and backward-induction construction follow Hull, Chapter 12 §§12.1–12.3 [1]. The node discount factors are model inputs. Calibration and short-rate dynamics are outside this lesson.

References

[1] Hull, Options, Futures, and Other Derivatives (8th ed., 2012).

Notation

$CF_{i+1,j}$ **Lattice next cash flow**. Signed cash paid at a successor node immediately before its continuation value is measured. *Units:* stated currency at the successor time.

$d_{i,j}$ **Lattice node discount factor**. One-period discount factor at a lattice node: currency at the node per one unit of currency at either successor node one step later. *Units:* current-node currency per next-time currency.

$q_{i,j}$ **Lattice node up weight**. Pricing probability assigned to the up successor, conditional on the current node; in a lattice with issuer default, also conditional on survival to the next lattice time. *Units:* probability between zero and one.

$V_{i,j}$ **Lattice node value**. Claim value at time row i and state node j obtained by one-period backward induction from its successor nodes, conditional on reaching that node under the input pricing lattice. *Units:* stated currency at the node time.

j **Lattice state index**. Integer label for one state node within a time row of a finite recombining lattice; a bookkeeping label under the stated successor ordering, not a probability or state value. *Units:* dimensionless integer index.

i **Lattice time index**. Integer label for one time row in a finite recombining valuation lattice; a bookkeeping label, not a model-year time or currency amount. *Units:* dimensionless integer index.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

Answers to the knowledge checks

Knowledge check 4.5.1: Multi-period lattice

1. 17.1 (accepted within ± 0.000001). The node value is $0.95 \times (0.6 \times 10 + 0.4 \times 30) = 17.1$.
2. 96 (accepted within ± 0.000001). Add the coupon before averaging: $0.96 \times [0.75 \times 95 + 0.25 \times 115] = 96$.