Chapter 6.1

Default, hazard, and survival

From *Credit Products Playground*, with changes after version 0.4.0, not yet released. Interactive version: https://danieltuzes.github.io/credit-and-derivatives/credit/default-hazard-and-survival/

6.1.1 What you will be able to do

After this lesson, you should be able to:

- interpret a survival probability without confusing it with a discount factor;
- calculate the default probability of one interval from the survival curve;
- explain what a constant hazard rate means in this toy model;
- calculate the survival probability from a constant hazard rate.

6.1.2 Survival probability: no default by time t

The survival-probability $S^{\mathbb{Q}}(0,t)$ is the probability, under the risk-neutral-probability-measure $\mathbb{Q}$, that the reference entity has not defaulted between valuation-time 0 and a future time-from-valuation t. With the default-time τ, the survival probability is defined as:

$$S^{\mathbb{Q}}(0,t) = \mathbb{Q}(\tau > t) \tag{6.1.1}$$

In this lesson, the survival probabilities are inputs: the lesson states their values and does not calculate them. In practice, a survival curve is fitted to market prices. The CDS lesson on the credit curve fits one to CDS quotes.

A survival probability is not a discount factor. A survival probability is the probability of a modeled event. A discount factor converts an amount paid at a future time to its value at time zero. A survival probability and a discount factor can have equal values and still mean different things.

We assume that the survival curve starts at one:

$$S(0,0) = 1 \tag{6.1.2}$$

We also assume that the survival curve does not increase with time. This lesson does not estimate survival probabilities from historical data, and it does not fit them to market quotes. Tuckman and Serrat introduce survival and default probabilities for CDS valuation[1]. Hull distinguishes

risk-neutral default probabilities, which are used in valuation, from real-world default probabilities [2].

6.1.3 Interval default probability: a decrease in survival

The local credit-period-index i selects one interval of the survival schedule, and the number-of-credit-periods n counts the intervals. Interval i starts at t_{i-1} and ends at the payment-time t_i, with $t_{i-1} < t_i$. The interval-default-probability of interval i is the decrease in survival probability over the interval:

$$\Delta q_i^{\mathbb{Q}} = S^{\mathbb{Q}}(0, t_{i-1}) - S^{\mathbb{Q}}(0, t_i) \tag{6.1.3}$$

The survival probability at the start of the interval comes first in the subtraction. Because the survival curve does not increase, this order gives $\Delta q_i \geq 0$. The sum of the interval default probabilities from time zero to t_n is the cumulative default probability by t_n:

$$\sum_{i=1}^{n} \Delta q_i = 1 - S(0, t_n) \tag{6.1.4}$$

This identity holds because the sum telescopes: each intermediate survival probability appears once with a plus sign and once with a minus sign. The identity uses two assumptions: $S(0, 0) = 1$, and at most one modeled default.

6.1.4 Constant hazard rate: an intensity, not a cumulative probability

In this lesson, the Constant hazard rate λ is constant. It is a conditional default intensity, measured per model-year. The hazard rate is not the probability of default by the end of one year. It is also not a discount rate. With the local credit-small-time-increment Δt, the hazard rate is defined as the limit of a conditional default probability per unit of time:

$$\lambda = \lim_{\Delta t \downarrow 0} \frac{\mathbb{Q}(t < \tau \leq t + \Delta t \mid \tau > t)}{\Delta t} \tag{6.1.5}$$

The probability in this definition is conditional on the event $\tau > t$. So the hazard rate at time t describes only the paths on which the reference entity has survived to t.

With a constant hazard rate, the survival probability is:

$$S^{\mathbb{Q}}(0, t) = \exp(-\lambda t) \tag{6.1.6}$$

So the cumulative default probability by t is $1 - S(0, t)$. This lesson does not introduce piecewise hazard rates, curve calibration, or a statistical estimator. Tuckman and Serrat derive the conditional intensity and the constant-hazard survival probability[3], and so does Hull [4].

Example 6.1.1: Survival and default probabilities

One interval drop

Suppose the one-year survival probability is $S(0, 1) = 0.97$. With $S(0, 0) = 1$, the interval default probability of the first year is:

$$\Delta q_1 = 1 - 0.97 = 0.03 = 3\%$$

The 3% is the probability of default during the first year. It is not discounted.

Three years at constant hazard

Let $\lambda = 0.02$ per model-year and $t = 3$ model-years. Substituting these values into (6.1.6) gives the three-year survival probability:

$$S(0,3) = \exp(-0.02 \times 3) = \exp(-0.06) \approx 0.941765$$

The cumulative default probability by three years is:

$$1 - S(0,3) \approx 0.058235 = 5.8235\%$$

The cumulative default probability is not $\lambda t = 6\%$. The product λt is the exponent in the survival probability. It is not the exact cumulative default probability.

Two unequal interval drops

Consider this survival curve:

Interval index i	Time t_i, model-years	Survival $S(0,t_i)$	Interval default Δq_i
0	0	1.00	—
1	1	0.97	0.03
2	2	0.92	0.05

The interval default probability of the second interval, 0.05, is larger than that of the first interval, 0.03, although both intervals are one model-year long. The cumulative two-year default probability is $0.03 + 0.05 = 0.08 = 1 - 0.92$.

6.1.5 Check your understanding

The assessment separates survival meaning, interval subtraction, hazard interpretation, and constant-hazard arithmetic. Each competency has a direct item and a transfer item.

Knowledge check 6.1.1: Default, hazard, and survival

1. A model gives S(0,3) = 0.92 for a name that has survived to valuation time 0. Which interpretation is correct?
   (a) The modeled probability of no default through year 3 is 92%
   (b) The modeled probability of default by year 3 is 92%
   (c) The three-year discount factor is necessarily 0.92
2. Under the same model and valuation time, name A has five-year survival probability 0.84 and name B has five-year survival probability 0.93. Which conclusion is supported?
   (a) Name A has the larger modeled probability of default by year 5
   (b) Name B has the larger modeled probability of default by year 5
   (c) The two names have the same modeled probability of default by year 5

3. A model gives two-year survival probability S(0,2) = 0.93. Enter the cumulative probability of default by year 2 as a decimal.

4. Viewed from valuation time 0, survival probabilities are S(0,1) = 0.97 and S(0,2) = 0.91. Enter the unconditional probability of default during the interval from year 1 to year 2 as a decimal.

5. In the lesson model, what does a hazard rate at time t describe?

   (a) Instantaneous default intensity conditional on survival to time t

   (b) Cumulative default probability from time 0 through time t

   (c) The interest rate used to discount every cash flow

6. Two names have survived to the same instant. Name A's current hazard rate is 2% per year and name B's is 5% per year. Over the next very short interval, which statement matches the hazard interpretation?

   (a) Name B has the higher modeled conditional default likelihood

   (b) Name A has the higher modeled conditional default likelihood

   (c) The rates alone prove both names have already defaulted

7. Under the constant-hazard exponential model, calculate the dimensionless three-year survival probability for an annual hazard rate of 2% (0.02 per year).

8. Under the constant-hazard exponential model, calculate the dimensionless survival probability over 18 months for an annual hazard rate of 3.2% (0.032 per year).

Answers: *end of the chapter.*

6.1.6 Model boundary and review note

This lesson uses a toy model with at most one default, a constant hazard rate, exact model-year times, and survival probabilities that are inputs. The model omits:

- calendar dates, day counts, and business-day adjustments;
- stochastic interest rates;
- dependence between interest rates and default;
- estimation and calibration.

The lesson makes no claim that pricing-model probabilities are forecasts of realized default frequencies.

All materially changed source locators, formulas, examples, notation, and answer keys remain 'draft' pending independent human review.

References

[1] Tuckman & Serrat, Fixed Income Securities: Tools for Today's Markets (4th ed., 2022). Ch. 14 §14.6, printed pp. 367–370, eqs. 14.4–14.7.

[2] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 23 §23.5, printed pp. 528–530.

[3] Tuckman & Serrat, Fixed Income Securities: Tools for Today's Markets (4th ed., 2022). Ch. 14 §14.6, printed pp. 367–368; Appendix A14.1, printed p. 505, eqs. A14.1–A14.2.

[4] Hull, Options, Futures, and Other Derivatives (8th ed., 2012). Ch. 23 §§23.2 and 23.5, printed pp. 522–523 and 528–530.

Notation

i **Credit period index**. Selects one interval in the lesson's ordered survival schedule. The index is bookkeeping and is not itself a time or probability.

$$i \in \{1, \dots, n\}$$

Δt **Credit small time increment**. Positive model-time interval used in the conditional definition of hazard. The hazard definition takes this increment down toward zero while holding the starting time fixed. *Units:* model-years.

$$\Delta t > 0$$

n **Number of credit periods**. Counts the intervals in the input survival schedule. It is a positive integer count for this finite toy model.

$$n \in \{1, 2, ...\}$$

$A(0,t)$ **Accumulation factor**. Grows one current unit over a stated future horizon under the selected rate model. *Units:* dimensionless currency-units per current currency-unit.

m **Compounding frequency**. Number of equal compounding periods per year under the stated rate convention, a positive integer fixed by the model convention. *Units:* compounding periods per year.

τ **Default time**. Random model time at which the reference entity first defaults. It is measured in model-years from valuation time under the stated risk-neutral model. *Units:* model-years after valuation time.

$$\tau > 0$$

$D(0,t)$ **Discount factor**. Converts one deterministic future unit into value at valuation time. *Units:* current currency-units per future currency-unit.

$\mathbb{E}$ **Expectation**. Operator returning the average of a random quantity, each outcome weighted by its probability under a stated measure; a superscript names that measure when more than one is in play. *Units:* value units.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

λ **Constant hazard rate**. Constant conditional default intensity used by the lesson's simplified exponential survival model; a risk-neutral pricing input conditional on survival to the current instant, not a cumulative probability. *Units:* decimal intensity per model-year.

Δq_i **Interval default probability**. Probability assigned by the model to first default during one stated time interval, conditional only through the input survival curve construction. *Units:* probability between zero and one.

$j^{(m)}$ **Nominal annual rate**. Annualized rate quote that must be paired with its compounding frequency. *Units:* decimal rate per year.

t_k **Payment time**. Time from the valuation date to one scheduled cash-flow event; a time coordinate, not a calendar date or payment index. *Units:* years from the valuation date.

r_m **Periodic rate**. Rate applied once in each compounding period under the stated convention, derived from the stated nominal annual quote in this model. *Units:* decimal per compounding period.

$\mathbb{Q}$ **Risk-neutral probability measure**. Gives model pricing weights under which discounted traded prices satisfy the martingale condition; it prices payoffs relative to a stated numeraire and is not a forecast of actual event frequencies. *Units:* dimensionless probability weights between zero and one.

$S(0,t)$ **Survival probability**. Probability, under the explicitly stated model measure, that no modeled default has occurred between valuation time and a stated future time. *Units:* probability between zero and one.

t **Time from valuation**. A time measured in years from the valuation time; a time coordinate, not a calendar date or a schedule index. With a row index it is a payment time. *Units:* years from the valuation date.

0 **Valuation time**. Common origin from which later model times and present values are measured. *Units:* years from the valuation date.

Answers to the knowledge checks

Knowledge check 6.1.1: Default, hazard, and survival

1. **(a)** The modeled probability of no default through year 3 is 92%. S(0,3) is the model's probability, conditional on survival at valuation time, that no default occurs through year 3.

2. **(a)** Name A has the larger modeled probability of default by year 5. Lower survival to the same horizon implies higher cumulative default probability to that horizon: 1 - 0.84 exceeds 1 - 0.93.

3. 0.07 (accepted within $\pm 1e-12$). Cumulative default probability is 1 - S(0,2) = 1 - 0.93 = 0.07.

4. 0.06 (accepted within $\pm 1e-12$). The interval default probability is S(0,1) - S(0,2) = 0.97 - 0.91 = 0.06.

5. **(a)** Instantaneous default intensity conditional on survival to time t. Hazard is an instantaneous conditional intensity with inverse-time units, not a cumulative probability or discount rate.

6. **(a)** Name B has the higher modeled conditional default likelihood. Conditional on both names having survived to the instant, the higher current hazard corresponds to greater modeled default likelihood over a sufficiently short next interval.

7. 0.941764533584249 (accepted within ± 0.000001). $S(0,3) = \exp(-0.02 \times 3) = \exp(-0.06) = 0.9417645336$.

8. 0.953133787077505 (accepted within ± 0.000001). Eighteen months is 1.5 years, so survival is $\exp(-0.032 \times 1.5) = \exp(-0.048) = 0.9531337871$.